JOURNAL OF ALGEBRA **92**, 44–80 (1985)

Maximal Subgroups of Finite Groups

M. ASCHBACHER AND L. SCOTT*,†

Department of Mathematics, California Institute of Technology, Pasadena, California 91125 and
**Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903*

Communicated by Walter Feit

Received February 15, 1983

What ingredients are necessary to describe all maximal subgroups of the general finite group G? This paper is concerned with providing such an analysis.

A good first reduction is to take into account the first isomorphism theorem, which tells us that the maximal subgroups containing a given normal subgroup N of G correspond, under the natural projection, to the maximal subgroups of the quotient group G/N. Let $\mathscr{M} = \mathscr{M}_G$ denote the collection of maximal subgroups of G, and let $\mathscr{M}^*$ be the subset of those $M \in \mathscr{M}$ with $\mathrm{Ker}_G(M) = 1$, where $\mathrm{Ker}_G(M)$ denotes the largest normal subgroup of G contained in M. Then the first isomorphism theorem allows us to identify $\mathscr{M}$ with the disjoint union $\coprod_{N \lhd G} \mathscr{M}^*_{G/N}$. Actually, what we really want to parameterize are the *conjugacy classes* of maximal subgroups, but this too works well: If $\mathscr{C} = \mathscr{C}_G$ denotes the set of G-conjugacy classes of elements of $\mathscr{M}$, and $\mathscr{C}^*$ is defined similarly, then we have

$$\mathscr{C}_G \cong \coprod_{N \lhd G} \mathscr{C}^*_{G/N}.$$

Hence our analysis is reduced to $\mathscr{M}^*$ and $\mathscr{C}^*$, rather than $\mathscr{M}$ and $\mathscr{C}$, if we assume a knowledge of the normal subgroup structure of G. The reduction is really even better than that, since often $\mathscr{M}_{G/N} = \phi$. For example, if G is a p-group, then $\mathscr{M}^*_{G/N} = \phi$ unless $|G/N| = p$.

Our main result is Theorem 1 below, which gives a general structural description of finite groups G with $\mathscr{M}^*_G$ nonempty, describes what η^*_G looks like, and determines $\mathscr{C}^*_G$ up to some difficult but well-defined problems:

Let G be a finite group with $L \leqslant G \leqslant \mathrm{Aut}(L)$ for some nonabelian simple group L, and let V be faithful irreducible G-module over some field of prime order.

† Both authors receive partial support from the National Science Foundation.

0021-8693/85 $3.00

(1) *Determine* $\mathscr{C}^*_{V.G} \cong H^1(G, V)$.

(2) *Determine* $\mathscr{C}^*_G$.

It is likely that a considerable knowledge of the irreducible modules for simple groups will enter into any final solution of either problem, cf. [4]. (For some corrections to [4] see the Appendix to this paper.)

The first cohomology group $H^1(G, V)$ of a group G on a G-module V has many interpretations, but the relevant one here is as the set of conjugacy classes of complements to V in the semidirect product $V.G$. This interpretation carries over to the case where V is nonabelian (cf. the discussion in Section 2) though $H^1(G, V)$ is no longer a group. In the process of obtaining Theorem 1 we also prove a result on nonabelian 1-cohomology of independent interest; that result is recorded in Theorem 2. Among other things, Theorem 2 gives the existence of some unusual subgroups in wreath products that are more or less invisible to standard techniques in group theory. Cf. the discussion of part (C)(1) of Theorem 1 at the end of the next section.

Finally, if G is a finite group and V a faithful irreducible module for G over a field of prime characteristic with $H^1(G, V) \neq 0$, then in Theorem 3 we determine the structure of the generalized Fitting subgroup of G and the representation of that subgroup on V.

Statement of the Main Results

We continue the notation introduced above with G a finite group. $\mathrm{Aut}(G)$, $\mathrm{Inn}(G)$, and $\mathrm{Out}(G)$ denote the group of automorphisms, inner automorphisms, and outer automorphisms of G, respectively. If H/K is a section of G then $\mathrm{Aut}_G(H/K)$ denotes the group of automorphisms of H/K induced in G; thus

$$\mathrm{Aut}_G(H/K) \cong (N_G(H) \cap N_G(K))/C_G(H/K)$$

and $C_G(H/K)$ consists of those $g \in N_G(H) \cap N_G(K)$ with $[H, g] \leqslant K$. $\mathrm{Out}_G(H/K)$ is the image of $\mathrm{Aut}_G(H/K)$ in $\mathrm{Out}(H/K)$.

We will often consider a group D with a given direct product decomposition $D = \prod_{i \in I} L_i$. A *diagonal subgroup* A of D (with respect to this decomposition) is a subgroup for which each projection $A \to L_i$ is injective; if these maps are in fact all isomorphisms then A is a *full diagonal* subgroup of D. $\mathscr{F}_D$ denotes the set of full diagonal subgroups of D and $\mathscr{D}_D$ denotes the D-conjugacy classes of such subgroups. For $J \subseteq I$ write $D_J = \prod_{j \in J} L_j$ and abbreviate $\mathscr{D}_{D_J} = \mathscr{D}_J$, etc., with $\mathscr{F} = \mathscr{F}_D$ and $\mathscr{D} = \mathscr{D}_D$ when no confusion can arise. Suppose G acts on D and permutes $\Delta = \{L_i : i \in I\}$. Then we denote by $\mathscr{D}(G)$ the G-stable classes in $\mathscr{D}$. Let $\mathscr{P}(G)$ denote the G-stable partitions Γ^G

of Δ and $\mathscr{P}^*(G)$ the maximal nontrivial G-stable partitions of Δ; that is, $\Gamma^G \in \mathscr{P}^*(G)$ if $\Gamma^G \in \mathscr{P}(G)$ and Γ is minimal subject to this constraint and to $|\Gamma| > 1$. Notice we do allow $\Gamma = \Delta$. For $L \in \Delta$, write $\mathscr{P}^*(G, L)$ for the collection of subsets Γ of Δ with $L \in \Gamma$ and $\Gamma^G \in \mathscr{P}^*(G)$.

Recall the *generalized Fitting subgroup* of G is the subgroup $F^*(G)$ generated by all subnormal nilpotent or quasisimple subgroups of G, with the latter subgroups called the components of G. A group is quasisimple if it is perfect and simple modulo its center. It turns out $F^*(G)$ is the central product of the Fitting subgroup of G and the components of G. An important special case of our analysis occurs when $D = F^*(G)$ is the direct product of the set Δ of all G-conjugates of some simple component L of G; in that event the notation and terminology of the last paragraph applies, and is used without comment.

We are now ready to state the main theorem.

THEOREM 1. *Let G be a finite group with $\mathscr{M}^*$ nonempty and set $D = F^*(G)$. Then one of the following holds:*

(A) *D is an elementary abelian p-group for some prime p, G acts irreducibly on D, and $\mathscr{M}^*$ is the set of complements to D in G. If $|\mathscr{C}^*| > 1$ then $F^*(G/D)$ is the direct product of the conjugates of a simple component L of G/D, $C_G(L)$ centralizes $U = [D, L]$, $\mathrm{Aut}_G(L)$ acts faithfully and irreducibly on U, and there is a natural bijection*

$$\mathscr{C}^* \cong H^1(\mathrm{Aut}_G(L), U).$$

(B) *D is the direct product of the G-conjugates of $L \times K$ where L and K are isomorphic nonconjugate simple components of G. Let $\mathscr{U}(L)$ consist of those U in K^G with $N_G(L) = N_G(U)$ and $D_{LU}(N_G(L)) \neq \varnothing$. Then $\mathscr{M}^*$ consists of those subgroups M of G such that $G = MD$ and $M \cap D$ is the direct product of the M-conjugates of a full diagonal subgroup of $LU(M)$ for some $U(M)$ in $\mathscr{U}(L)$. The map $M^G \mapsto (M \cap LU(M))^D$ gives a bijection*

$$\mathscr{C}^* \cong \coprod_{U \in \mathscr{U}(L)} \mathscr{D}_{LU}(N_G(LU)).$$

Indeed $\mathscr{C}^* \cong \coprod_{U \in \mathscr{U}(L)} C_{\mathrm{Out}(L)}(\mathrm{Out}_G(L))$.

(C) *D is the direct product of the set Δ of G-conjugates of some simple component L of G. The set $\mathscr{M}^*$ is the disjoint union of G-stable subsets $\mathscr{M}_i^*$, $1 \leqslant_i \leqslant 3$, corresponding to the three cases below, and $\mathscr{C}^*$ decomposes accordingly. Specifically*

$$\mathscr{M}_1^* = \{N \in \mathscr{M}^* : \mathrm{Aut}_N(L) = \mathrm{Aut}_G(L) \textit{ and } N \cap D = 1\}$$

$$\mathscr{M}_2^* = \{N \in \mathscr{M}^* : \mathrm{Aut}_N(L) = \mathrm{Aut}_G(L) \textit{ and } N \cap D \neq 1\}$$

$$\mathscr{M}_3^* = \{N \in \mathscr{M}^* : \mathrm{Aut}_N(L) \textit{ is maximal in } \mathrm{Aut}_G(L)\}$$

leading to the following three cases:

(1) *Let* $\mathscr{M}_1'$ *consist of those complements* M *to* D *with* $\mathrm{Inn}(L) \leqslant \mathrm{Aut}_M(L)$. *Then* $\mathscr{M}_1^*$ *consists of those members of* $\mathscr{M}_1'$ *contained in no member of* $\mathscr{M}_2^*$. *Moreover there is a natural bijection between the collection* $\mathscr{C}_1'$ *of orbits of* G *on* $\mathscr{M}_1'$ *and the set of* D*-classes of complements* X *to* $D/C_D(L)$ *in* $N_G(L)/C_D(L)$ *with* $\mathrm{Inn}(L) \leqslant \mathrm{Aut}_X(L)$.

Theorem 4 *gives a necessary and sufficient condition for a member of* $\mathscr{M}_1'$ *to be contained in* $\mathscr{M}_1^*$ *and gives a parametrization of* $\mathscr{C}_1^*$.

(2) $\mathscr{M}_2^*$ *consists of those subgroups* M *of* G *such that* $G = MD$, $M \cap D$ *is the direct product of the* M*-conjugates of* $M \cap D_\Gamma \in \mathscr{F}_\Gamma$, *for some* Γ *with* $\Gamma^G \in \mathscr{P}^*(G)$. *The map* $M^G \mapsto (M \cap D_\Gamma)^{D_\Gamma}$ *gives a bijection*

$$\mathscr{C}_2^* \cong \coprod_{\Gamma \in \mathscr{P}^*(G,L)} \mathscr{D}_{D_\Gamma}(N_G(D_\Gamma)).$$

Theorem 5, *applied to the various* D_Γ, *makes this parametrization explicit.*

(3) $\mathscr{M}_3^*$ *consists of those subgroups* M *of* G *such that* $G = MD$, $M \cap D$ *is the direct product of the* M*-conjugates of* $M \cap L$, $\mathrm{Aut}_M(L)$ *is a maximal subgroup of* $\mathrm{Aut}_G(L)$ *which does not contain* $\mathrm{Inn}(L)$, *and* $\mathrm{Aut}_M(L) \cap \mathrm{Inn}(L) = \mathrm{Aut}_{M \cap L}(L)$. *The map* $M^G \to (\mathrm{Aut}_M(L))^{\mathrm{Aut}_G(L)}$ *gives a bijection*

$$\mathscr{C}_3^* \cong \mathscr{C}^*_{\mathrm{Aut}_G(L)}.$$

THEOREM 2. *Let* X *be a group containing a normal subgroup* D *which is the direct product of the* X*-conjugates of some subgroup* L. *Let* $D' = \langle L^X - \{L\} \rangle$ *and if* T *is a complement to* D *in* X *define* $\mu(T) = D'N_T(L)/D'$. *Then* μ *is a surjective map from the set of all complements to* D *in* X *onto the set of all complements to* $D/D' \cong L$ *in* $N_X(L)/D'$, *and* μ *induces a bijection*

$$T^X \to \mu(T)^L$$

of conjugacy classes of complements.

An auxiliary result (3.6) *also describes which complements to* D *in* $N_X(L)$ *have the form* $N_T(L)$ *for some complement* T *to* D *in* X.

It was recently pointed out to us that on the one hand there is some overlap between Theorems 1 and 2 and some work of Gross and Kovacs in [7, 8], and on the other that Proposition 1.29 in [6] is the split case of Theorem 2.

THEOREM 3. *Let* G *be a finite group,* p *a prime,* K *a field of characteristic* p, *and* V *a faithful irreducible* KG*-module such that* $H^1(G, V) \neq 0$. *Then*

(1) $F^*(G)$ *is the direct product of the* G*-conjugates of a simple component* L *of* G *of order divisible by* p.

(2) *V is the direct sum of the G-conjugates of $U = [V, L]$.*

(3) *$N_G(L)$ acts irreducibly on U with $C_G(L) = C_G(U)$.*

(4) $H^1(G, V) \cong H^1(\mathrm{Aut}_G(L), U)$.

(5) $\dim(H^1(G, V)) \leqslant \dim(H^1(L, E))$ *for each nontrivial irreducible KL-submodule E of V.*

THEOREM 4. *Assume the hypothesis and notation of Theorem 1, and assume case C of Theorem 1 holds. Let $\mathscr{E}$ be the set of normal subgroups E of $N_G(L)$ such that*

(i) *$D \leqslant E \leqslant LC_G(L)$ with $DC_G(L)/E \cong L$, and*

(ii) *$\mathscr{D}_{((C_G(L)/C_E(L)) \times L)}(N_G(L))$ is nonempty, and*

(iii) *for each $\Gamma \in \mathscr{P}^*(G, L)$ with $\mathscr{D}_{D_\Gamma}(N_G(D_\Gamma))$ nonempty, the following two conditions hold:*

(a) *either E or $DC_G(L)$ is not normal in $N_G(D_\Gamma)$, and*

(b) *the section $N_G(L)/E$ has no normal complement in $N_G(D_\Gamma)$.*

Then we have

(1) *$\mathscr{M}_1^*$ consists of those $M \in \mathscr{M}_1'$ such that $DC_M(L) \in \mathscr{E}$.*

(2) $\mathscr{C}_1^* \cong \mathscr{E} \times C_{\mathrm{Out}(L)}(\mathrm{Out}_G(L))$.

By definition if $A \trianglelefteq B \leqslant C$ are groups then a complement *to the section A/B in C is a subgroup N of C with $BN = C$ and $B \cap N = A$.*

Lemma 7.30 gives another parameterization of $\mathscr{C}_1^*$ in terms of $N_G(L)$-invariant L-orbits of homomorphisms from $DC_G(L)/D$ to L. Similarly Theorem 5 in Section 5 gives a bijection between $\mathscr{D}(G)$ and the set of homomorphisms from G into $\mathrm{Out}(L)$ extending the conjugation map from $N_G(L)$ to $\mathrm{Out}(L)$ (in the event $\mathscr{D}(G)$ is nonempty). The latter result shows $|\mathscr{C}_2^*| \leqslant |\mathscr{P}^*(G, L)|\,|\mathrm{Out}(L)|$, for example (cf. 5.15 or 6.4.3.)

The question of when $\mathscr{C}^*$ is nonempty in case A or B and when $\mathscr{C}_2^*$ is nonempty in case C are left open. The first question simply asks if a certain group extension splits, and the other two have a similar flavor.

The maximal subgroups in part (C)(1) seem particularly interesting; each is a complement to $D = F^*(G)$ in G, although very complement need not be maximal. Theorem 2 gives a bijection between the complements to D in G and the complements to L in $N_G(L)/C_D(L)$, and Theorem 4 and Lemma 7.30 supply a means for deciding maximality. Lemmas 7.1 and 7.2 give ad hoc methods for deciding whether $\mathscr{C}_1^*$ is nonempty. To illustrate the theory, consider the wreath product $G = A_5 \operatorname{wr} A_6$. Then $N_G(L)/C_D(L) \cong A_5 \times A_5$, so a full diagonal subgroup X of $N_G(L)/C_D(L)$ is a complement to L. Hence by Theorem 2 there exists a complement M to D in G with $N_M(L)\, C_D(L)/$

$C_D(L)=X$, and by 7.2 M is maximal in G. Carlos Scoppola supplies an alternate argument for the existence of at least one nonstandard complement in $G=A_5 \operatorname{wr} A_6$. Namely, A_6 has a transitive representation on 30 letters with 6 blocks of imprimitivity of size 5, and hence is embedded in a copy of G in S_{30} preserving this set of blocks.[1]

1. Diagonal Subgroups

Let L be a group, $\mathscr{F}=(\alpha_i: L\to L_i: i\in I)$ a family of isomorphisms, and $D=\prod_{i\in I} L_i$ the direct product of the groups L_i, $i\in I$. For $x\in L$ define the *diagonal* of $\mathscr{F}$ to be

$$\operatorname{diag}(\mathscr{F})=\left\{\prod_{i\in I} x\alpha_i : x\in L\right\}.$$

Let $\operatorname{Aut}_I(D)$ be the subgroup of $\operatorname{Aut}(D)$ permuting $\Delta=\{L_i: i\in I\}$.

(1.1) *Let S be the symmetric group on I, $L_i^*=\operatorname{Aut}(L_i)$, and $D^*=\prod_{i\in I} L_i^*$. α_i induces the isomorphism*

$$\alpha_i^*: L^*=\operatorname{Aut}(L)\to L_i^*$$
$$\beta\to\beta^{\alpha_i}$$

and if we set $\mathscr{F}^*=(\alpha_i^*: i\in I)$ and let S act on D via

$$s\colon \prod_{i\in I} x_i\alpha_i\to\prod_{i\in I} x_{is}\alpha_i,\qquad s\in S, x_i\in L$$

then

(1) $A=\operatorname{Aut}_I(D)$ *is the semidirect product of D^* and S.*

(2) *S acts on D^* via*

$$s\colon \prod_{i\in I} x_i^*\alpha_i^*\to\prod_{i\in I} x_{is}^*\alpha_i^*,\qquad s\in S,\ x_i^*\in L^*.$$

(3) $N_A(\operatorname{diag}(\mathscr{F}))=S\times\operatorname{diag}(\mathscr{F}^*)\cong S\times\operatorname{Aut}(L)$.

Proof. A_Δ is isomorphic to a subgroup of D^*, while $D^*\leqslant A_\Delta$, so $D^*=A_\Delta\trianglelefteq A$. Thus D^*S is subgroup of A. Also $A^\Delta\leqslant\operatorname{Sym}(\Delta)=S^\Delta$, so $A^\Delta=S^\Delta$. Hence $A=SA_\Delta=SD^*$, and as $S_\Delta=1$, the product is semidirect. So (1) holds.

For $i\in I$ let $D_i=\prod_{j\neq i} L_i$. Then $L_i^*=C_{D^*}(D_i)$ so for $s\in S$, $(L_i^*)^s=L_{is}^*$, and in particular $U=N_S(L_i)=N_S(L_i^*)$. Further $[U,L_i^*]\leqslant C_A(L_i)\cap C_A(D_i)=C_A(D)=1$. Therefore (2) holds.

[1] *Note added in proof.* K. Gruenberg and L. Kovacs have informed the authors that they independently obtained a result similar to Theorem 3.

Let $B = \mathrm{diag}(\mathscr{F})$ and $B^* = \mathrm{diag}(\mathscr{F}^*)$. Evidently $B^*S = B^* \times S \leqslant N_A(B) = M$. $M^\Delta \leqslant A^\Delta = S^\Delta$, so $M = SM_\Delta$. $B^* \leqslant M_\Delta$ with $\mathrm{Aut}(B) \cong B^*$ and B^* is faithful on B, so $M_\Delta = B^*C(B)_\Delta$. Finally, if $g \in C(B)_\Delta$ then g centralizes $x\alpha$ and hence also its projection $x\alpha_i$ on L_i for each $x \in L$ and $i \in I$. Thus $g \in C_A(D) = 1$, so $C(B)_\Delta = 1$ and $M = SB^*$.

Remark. In Section 3 we discuss wreath products. Notice that from that discussion and Lemma 1.1, $\mathrm{Aut}_I(D)$ is just the wreath product $L^* \mathrm{wr}_I S$ where S is the symmetric group on I and $L^* = \mathrm{Aut}(L)$.

A *diagonal subgroup* of D is a subgroup X of D such that each projection $\pi_i: X \to L_i$, $i \in I$, is an injection. A *full diagonal subgroup* is a diagonal subgroup for which each projection is an isomorphism.

(1.2) *Let $i' \in I$, $L = L_{i'}$, $\mathscr{D}$ the set of full diagonal subgroups of D, and $\mathscr{A}$ the set of families $\mathscr{F} = (\alpha_i: L \to L_i: i \in I)$ of isomorphisms with $\alpha_{i'} = 1$. Then*

(1) *The map $\mathscr{F} \to \mathrm{diag}(\mathscr{F})$ is a bijection of $\mathscr{A}$ with $\mathscr{D}$.*

(2) *The map in (1) commutes with the actions of $C_{D^*}(L)$ on $\mathscr{D}$ and on $\mathscr{A}$ by right multiplication. D^* is transitive on $\mathscr{D}$ and $C_{D^*}(L)$ is regular on $\mathscr{D}$.*

(3) *If L and I are finite then $|\mathscr{D}| = |\mathrm{Aut}(L)|^n$, where $|I| = n+1$.*

Proof. For $X \in \mathscr{D}$ define $\alpha_i^X = \pi_{i'}^{-1}\pi_i: L \to L_i$ and $\mathscr{F}'(X) = (\alpha_i^X: i \in I)$. Then $\mathscr{F}'(X) \in \mathscr{A}$ with $X = \mathrm{diag}(\mathscr{F}'(X))$. Conversely if $\mathscr{F} \in \mathscr{D}$ then $\mathscr{F}'(\mathrm{diag}(\mathscr{F})) = \mathscr{F}$, so (1) holds.

Evidently the map in (1) commutes with the actions of $C_{D^*}(L)$, and $C_{D^*}(L)$ is regular on $\mathscr{A}$, so (2) holds. Assume L and I are finite with $|I| = n+1$. As $C_{D^*}(L)$ is regular on $\mathscr{A}$, $|\mathscr{D}| = |\mathscr{A}| = |C_{D^*}(L)| = |\mathrm{Aut}(L)|^n$, so (3) holds.

(1.3) *Let $G \leqslant \mathrm{Aut}_I(D)$ with G transitive on Δ. Let $X \leqslant L = L_{i'}$ with $X^{N_G(L)} = X^L$. Let Ω be the set of subgroups Y of D such that $Y = \prod_{i \in I} X_i$ for some $X_i \in X^G \cap L_i$. Then D is transitive on Ω if I and L are finite.*

Proof. Let $\mathscr{O} = X^L$. As $\mathscr{O}$ is $N_G(L)$-invariant, $\mathscr{O}^G$ is of order $n = |G: N_G(L)| = |I|$. Hence

$$\mathscr{O}^G = \{X_i^{L_i}: i \in I\}$$

for some $X_i \in X^G \cap L_i$. Now $|\Omega| = |\mathscr{O}|^n = |L: N_L(X)|^n = |D: N_D(Y)|$ for $Y = \prod_{i \in I} X_i \in \Omega$. So D is transitive on Ω.

In the remainder of this section we assume L to be a finite nonabelian simple group and assume I is finite. Then we may identify L_i with its group $\mathrm{Inn}(L_i)$ of inner automorphisms and hence identify D with the normal subgroup $\mathrm{Inn}(D)$ of $A = \mathrm{Aut}_I(D)$. Moreover $\mathrm{Aut}(D)$ permutes the components L_i, $i \in I$, of D, so $A = \mathrm{Aut}(D)$.

Let $\mathscr{P}$ be the set of all partitions of Δ. $P \in \mathscr{P}$ is *nontrivial* if $|\Gamma| > 1$ for some $\Gamma \in P$. If $G \leqslant A$ then G permutes $\mathscr{P}$ in an obvious fashion. Partially order $\mathscr{P}$ by $P \leqslant Q$ if P is a refinement of Q. The *minimal nontrivial G-invariant partitions* (mentioned in the statement of Theorem 1) are the minimal members of the set of all nontrivial G-invariant partitions under this partial order.

For $\Gamma \subseteq \Delta$ let $D_\Gamma = \prod_{K \in \Gamma} K$. Let $\mathscr{B}$ be the set of tuples $(H_\Gamma : \Gamma \in P)$ as P varies over $\mathscr{P}$ and H_Γ varies over the set of full diagonal subgroups of D_Γ. Partially order $\mathscr{B}$ by $(H_\Gamma : \Gamma \in P) \leqslant (\tilde{H}_\Omega : \Omega \in \tilde{P})$ if $\tilde{P} \leqslant P$ and H_Γ is a full diagonal subgroup of $\prod_{\Omega \in \tilde{P}(\Gamma)} \tilde{H}$ (with respect to that decomposition), where $\tilde{P}(\Gamma)$ consists of those members of $\tilde{P}$ contained in Γ.

Let $\mathscr{H}$ be the collection of subgroups H of D such that the projections $\pi_i : H \to L_i$ are surjections for each $i \in I$. Partially order $\mathscr{H}$ by inclusion.

We now state two lemmas from [4]. For completeness we include proofs of these lemmas.

(1.4) *Let $H \in \mathscr{H}$. Then there exists $P \in \mathscr{P}$ such that $H = \prod_{\Gamma \in P} H\pi_\Gamma$ is the direct product of the full diagonal subgroups $H\pi_\Gamma$ of D_Γ, where $\pi_\Gamma : H \to D_\Gamma$ is the projection map with respect to the decomposition $D = \Pi_{\Gamma \in P} D_\Gamma$.*

Proof. Let $\Gamma \subseteq \Delta$ be minimal subject to $K = H \cap B \neq 1$, where $B = D_\Gamma$. By minimality of Γ the projection $\pi_i : K \to L_i$ is nontrivial for each $L_i \in \Gamma$. As $B \trianglelefteq D$, $K = H \cap B \trianglelefteq H$, so $1 \neq K\pi_i \trianglelefteq H\pi_i = L_i$. As L_i is simple we conclude $\pi_i : K \to L_i$ is a surjection for each $L_i \in \Gamma$. Next $\ker(\pi_i) \leqslant \prod_{J \in \Gamma - \{L_i\}} J$, so by minimality of Γ, $\ker(\pi_i) = 1$. Thus π_i is an isomorphism and K is a full diagonal subgroup of B.

Let $E = D_{\Delta - \Gamma}$, so that $D = E \times B$. Let $\pi : H \to B$ be the projection with respect to this decomposition. Again $K = K\pi \trianglelefteq H\pi$. But as K is a full diagonal subgroup of B, 1.2 and 1.1.3 imply $K = N_B(K)$, so $H\pi = K$. Thus $H = K \times C_H(K)$ with $C_H(K) = \ker(\pi) = H \cap E$. Finally, for $L_i \in \Delta - \Gamma$, $L_i = H\pi_i = (H \cap E)\pi_i$, so the lemma holds by induction on the order of I.

(1.5) (1) *The map $(H_\Gamma : \Gamma \in P) \to \prod_{\Gamma \in P} H_\Gamma$ is an isomorphism of $\mathscr{B}$ and $\mathscr{H}$ as partially ordered sets.*

(2) *If $H \in \mathscr{H}$ and $H \leqslant X \leqslant D$ then $X \in \mathscr{H}$.*

Proof. If $H \in \mathscr{H}$ and $H \leqslant X \leqslant D$, then $L_i = H\pi_i \leqslant X\pi_i$, so $X \in \mathscr{H}$ and (2) holds.

Let φ be the map defined in (1). Evidently φ maps $\mathscr{B}$ into $\mathscr{H}$ and preserves the partial order. By 1.4, φ is a surjection. As the components of $\prod_{\Gamma \in P} H_\Gamma$ are the groups H_Γ, $\Gamma \in P$, φ is an injection. Suppose $H = \prod_{\Gamma \in P} H_\Gamma$

is a subgroup of $\prod_{\Omega\in\tilde{P}}\tilde{H}_\Omega$. Let $\pi_\Omega: H\to\tilde{H}_\Omega$ and $\pi_i: D\to L_i$, $i\in\Omega$ be the projections. $L_i=H\pi_i=H\pi_\Omega\pi_i\leqslant(\tilde{H}_\Omega)\pi_i$ and π_i: $\tilde{H}_\Omega\to L_i$ is an isomorphism, so π_Ω is a surjection. Thus by 1.4 there is a partition P' of $\tilde{P}$ such that for each $\Gamma\in P$, H_Γ is a full diagonal subgroup of $\prod_{\Omega\in\theta}\tilde{H}_\Omega$ for some $\theta\in P'$. Then $\Gamma=\bigcup_{\Omega\in\theta}\Omega$, so $\tilde{P}$ is a refinement of P and $(H_\Gamma:\Gamma\in P)\leqslant(\tilde{H}_\Omega:\Omega\in\tilde{P})$ as desired.

(1.6) *Assume $G\leqslant A=\mathrm{Aut}(D)$ and $M\leqslant G$ with $G=MD$, M is transitive on Δ, and $H=M\cap D=\prod_{\Gamma\in P}H_\Gamma\in\mathscr{H}$, for some $(H_\Gamma:\Gamma\in P)\in\mathscr{B}$. Then*

(1) $M=N_G(H)$.

(2) *P is G-invariant.*

(3) *For each $\Gamma\in P$, $(H_\Gamma)^{D_\Gamma}$ is a $N_G(D_\Gamma)$-invariant class of full diagonal subgroups of D_Γ.*

(4) *Assume $\tilde{P}$ is a G-invariant refinement of P and for $\Omega\in\tilde{P}$ let $\tilde{H}_\Omega$ be the projection of H on D_Ω. Then $\tilde{H}=\prod_{\Omega\in\tilde{P}}\tilde{H}_\Omega$ is an M-invariant member of $\mathscr{H}$ containing H and $(\tilde{H}_\Omega)^{D_\Omega}$ is $N_G(D_\Omega)$-invariant.*

(5) *$M\in\mathscr{M}_G$ if and only if P is a minimal nontrivial G-invariant partition of Δ.*

Proof. As D is trivial on Δ and $G=MD$, $\tilde{P}\in\mathscr{P}$ is G-invariant precisely when $\tilde{P}$ is M-invariant. As H_Γ, $\Gamma\in P$, are the components of H and $H=M\cap D\trianglelefteq M$, P is M-invariant. Thus (2) holds. Also $N_G(H)=MN_D(H)$ and $N_D(H)=\prod_{\Gamma\in P}(N(H_\Gamma)\cap D_\Gamma)$. By 1.2 and 1.1.3, $H_\Gamma=N(H_\Gamma)\cap D_\Gamma$, so (1) holds. Also for $\Gamma\in P$, $N_G(D_\Gamma)=DN_M(D_\Gamma)$ and $N_M(D_\Gamma)\leqslant N(H_\Gamma)$. Hence $(H_\Gamma)^{D_\Gamma}$ is $N_G(D_\Gamma)$-invariant, so (3) holds.

Assume the hypothesis and notation of (4). For $\Omega\in\tilde{P}$, $N_M(D_\Omega)$ acts on the projection $\tilde{H}_\Omega$ of H on D_Ω and for $g\in M$, $\Omega^g\in\tilde{P}$, and $\tilde{H}_{\Omega^g}=(\tilde{H}_\Omega)^g$. Hence $M\leqslant N_G(\tilde{H})$. Of course $H\leqslant\tilde{H}$, so $m\tilde{H}\cap D=\tilde{H}$ and the last part of (4) follows from (3).

If $M\in\mathscr{M}$ then by (4), P is a minimal nontrivial G-invariant partition of Δ. Conversely assume P is maximal and let $M<Y\leqslant G$. Then $Y=MX$, where $H\leqslant X=Y\cap D$. By 1.5, $X=\prod_{\Omega\in\tilde{P}}X_\Omega$ for some refinement $\tilde{P}$ of P. As $M\leqslant N(X)$, $\tilde{P}$ is G-invariant, so by minimality of P, $\tilde{P}$ is trivial. Hence $X=D$ and $Y=MX=G$. That is, $M\in\mathscr{M}$.

2. 1-Cohomology

In this section we recall some facts about 1-cohomology.

Let G be a group acting on a group A. In this section we write our actions on the left. For $x,y\in G$, ${}^xy=y^{x^{-1}}=xyx^{-1}$. $A.G$ denotes the semidirect product of A and G. The set $\Gamma(G,A)$ of *cocycles* consists of the functions $\gamma: G \to A$ satisfying the *cocycle condition*

$$\gamma_{gh} = \gamma_g \cdot {}^g\gamma_h$$

for all $g, h \in G$. γ_g denotes the image of g in A under γ, and for $a \in A$, ${}^g a$ is the image of a under g.

Given a cocycle γ, let $\gamma^*: G \to A.G$ be defined by $\gamma^*(g) = \gamma_g g$. The cocycle condition is equivalent to the assertion that γ^* is a homomorphism. Indeed the map $\gamma \to \gamma^*(G)$ is a bijection of $\Gamma(G,A)$ with the set of complements to A in $A.G$. Moreover if $a \in A$, then ${}^a(\gamma^*(G)) = \beta^*(G)$, where β is the cycle

$$\beta: g \to a \cdot \gamma_g \cdot {}^g a^{-1}.$$

Cocycles γ and γ' are defined to be *cohomologous* if there exists $a \in A$ with $\gamma'_g = a_{\gamma_g} {}^g a^{-1}$ for all g in G. Cohomology is an equivalence relation and we write $[\gamma]$ for the cohomology class of γ, and write $H^1(G,A)$ for the set of cohomology classes. From the discussion above the map $[\gamma] \to {}^A\gamma^*(G)$ defines a bijection between $H^1(G,A)$ and the set of conjugacy classes of complements to A in $A.G$. The *trivial class* in $H^1(G,A)$ is the class of the trivial cocycle mapping each element of G to 1. This cocycle corresponds to the "standard copy" of G in $A.G$.

(2.1) *If G' is a complement to A in $A.G$ then $H^1(G,A) \cong H^1(G',A)$. This assertion is immediate from the bijections of $H^1(G,A)$ and $H^1(G',A)$ with the classes of complements to A in $A.G$.*

A acts on $\mathrm{Hom}(G,A)$ *via*:

$$({}^a\varphi)(g) = {}^a(\varphi(g)), \qquad a \in A,\ g \in G,\ \varphi \in \mathrm{Hom}(G,A).$$

Denote by $\mathscr{H}\mathrm{om}(G,A)$ *the set of orbits of A on* $\mathrm{Hom}(G,A)$ *under this action.*

(2.2) *Assume G acts by inner automorphisms on A in the sense that there exists $i \in \mathrm{Hom}(G,A)$ with ${}^g a = {}^{i(g)}a$ for each $a \in A$, $g \in G$. For $\gamma \in \Gamma(G,A)$ define*

$$\pi(\gamma): \to A$$
$$g \mapsto \gamma_g \cdot i(g).$$

Then $\pi(\gamma) \in \mathrm{Hom}(G,A)$ and the map

$$[\gamma] \mapsto {}^A(\pi(\gamma))$$

defines a bijection $H^1(G,A) \cong \mathscr{H}\mathrm{om}(G,A)$.

Proof. A straightforward calculation shows $\pi(\gamma) \in \mathrm{Hom}(G, A)$ and the map $[\gamma] \to {}^{A}(\pi(\gamma))$ is a well-defined injection. If $\alpha \in \mathrm{Hom}(G, A)$ then define $\gamma: G \to A$ by $\gamma g = \alpha(g)\, i(g)^{-1}$. Then $\gamma \in \Gamma(G, A)$, so the map is a surjection.

Some more notation: Let $\mathscr{H}\mathrm{om}^*(G, A)$ denote the set of orbits of surjective homomorphisms. If Z is a group acting on G and A then Z acts on $\mathrm{Hom}(G, A)$ via $({}^{z}\varphi)(g) = {}^{z}(\varphi(g^{z}))$, $z \in Z$, $g \in G$, $\varphi \in \mathrm{Hom}(G, A)$. Further this action induces an action of Z on $\mathscr{H}\mathrm{om}^*(G, A)$ defined by ${}^{z}(\varphi^{A}) = ({}^{z}\varphi)^{A}$, $z \in Z$, $\varphi^{A} \in \mathscr{H}\mathrm{om}(G, A)$. Denote by $\mathscr{H}\mathrm{om}^*_Z(G, A)$ the fixed points of Z on $\mathscr{H}\mathrm{om}^*(G, A)$ under this action.

(2.3) *Let A be a nonabelian simple group normal in G and let $\mathscr{A}$ denote the set of conjugacy classes M^G of complements to A in G such that $\mathrm{Inn}(A) \leqslant \mathrm{Aut}_M(A)$. For $M^G \in \mathscr{A}$ define η_M to be the homomorphism from $C_G(A)$ to A mapping an element to the inverse of its projection on A with respect to the decomposition $G = A.M$. Then the map $\eta: M^G \to (\eta_M)^A$ defines a bijection $\mathscr{A} \cong \mathscr{H}\mathrm{om}^*_G(C_G(A), A)$.*

Proof. Let $K = C_G(A)$ and notice $\mathrm{Hom}(K, A)$, $\mathscr{H}\mathrm{om}(K, A) = H^1(K, A)$, and $\mathscr{H}\mathrm{om}^*(K, A)$ consists of the orbits φ^A such that $\mathrm{Inn}(A) = \mathrm{Aut}_{\varphi^*(K)}(A)$. Moreover the map $M^G \mapsto (M \cap KA)^G$ defines a bijection between $\mathscr{A}$ and the G-invariant classes of complements $\varphi^*(K)^A$ with $\varphi^A \in \mathscr{H}\mathrm{om}^*(K, A)$. Hence to complete the proof we must show $\varphi^A \in \mathscr{H}\mathrm{om}^*(K, A)$ is a fixed point of G if and only if $\varphi^*(K)^A$ is G-invariant.

If $\varphi^*(K)^A$ is G-invariant then $G = N_G(\varphi^*(K))A$, and $N_G(\varphi^*(K))$ commutes with the map φ, so that φ^A is a fixed point of G. Conversely if φ^A is a fixed point, then as the action of A on $\mathrm{Hom}(K, A)$ is the restriction to A of the action of G on $\mathrm{Hom}(K, A)$, $G = AM$, where M is the stabilizer of φ in G, and $\varphi^*(K) = M \cap KA \lhd M$, so that $\varphi^*(K)^A$ is G-invariant.

(2.4) *Assume the hypothesis and notation of* 2.3 *with $K = C_G(A) \cong A$ and $\mathscr{A}$ nonempty. Then $\mathscr{A} \cong C_{\mathrm{Out}(A)}(\mathrm{Out}_G(A))$.*

Proof. As $\mathscr{A}$ is nonempty, we saw in the proof of 2.3 that there exists an isomorphism $\varphi: K \to A$ such that $G = AM$, where M is the stabilizer of φ in G. Let $g \to \alpha_g$ and $g \to \beta_g$ be the conjugation maps of G into $\mathrm{Aut}(A)$ and $\mathrm{Aut}(K)$, respectively. $\mathrm{Hom}^*(K, A) = \{\alpha \circ \varphi : \alpha \in \mathrm{Aut}(A)\}$ and for $\psi \in \mathrm{Hom}^*(K, A)$, $g \in G$, and $k \in K$, $({}^{g}\psi)(k) = {}^{g}(\psi(k^{g})) = (\alpha_g \circ \psi \circ \beta_g^{-1})(k)$, so ${}^{g}\psi = \alpha_g \circ \psi \circ \beta_g^{-1}$. As $\beta_g = 1$ for $g \in A$, $\psi^A = \mathrm{Inn}(A)\psi$, so $\pi: \psi^A \to \mathrm{Inn}(A)\psi$ is a bijection of $\mathscr{H}\mathrm{om}^*(K, A)$ with $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$. Also for $g \in M$, ${}^{g}\varphi = \varphi$, so $\varphi \circ \beta_g^{-1} = \alpha_g^{-1} \circ \varphi$, and thus writing $\psi = \alpha \circ \varphi$, ${}^{g}\psi = \alpha_g \circ \psi \circ \beta_g^{-1} = ({}^{(\alpha_g)}\alpha) \circ \varphi$, so π defines an M-isomorphism, where M acts on $\mathrm{Out}(A)$ by conjugation. As A is in the kernel of both actions, π is also a G-isomorphism so $\mathscr{A}$ is isomorphic to the fixed point set $C_{\mathrm{Out}(A)}(\mathrm{Out}_G(A))$ of G acting on $\mathrm{Out}(A)$ by conjugation.

3. The Proof of Theorem 2

In this section, actions of groups on sets will be on the left. Given group elements x and y we write ${}^xy$ for $y^{x^{-1}} = xyx^{-1}$. If S is a group acting on a group N, the semidirect product will be denoted by $N.S$. If S acts on a set Ω and L is a group, the *wreath product* $L \operatorname{wr} S = L \operatorname{wr}_\Omega S$ is the semidirect product $D.S$ where $D = \prod_\Omega L$ consists of all functions $f\colon \Omega \to L$ and S acts on D via

$$({}^sf)(\alpha^s) = f(\alpha^s), \qquad \alpha \in \Omega,\ s \in S \tag{$*$}$$

where by definition $\alpha^s = s_\alpha^{-1}$.

When Ω is the set S/U of left cosets tU of a subgroup U of S, it will often be convenient to view the functions in $\prod_\Omega L$ as defined on S, but constant on the cosets of U. That is, we let S act on itself by left multiplication and embed $L \operatorname{wr}_\Omega S$ in $L \operatorname{wr}_S S$ via the injection

$$L \operatorname{wr}_\Omega S \to L \operatorname{wr}_S S$$

$$fs \mapsto f'_s$$

where $f'\colon S \to L$ is defined by $f'(tu) = f(tU)$ for $t \in S$ and $u \in U$. Subject to this identification, Eq. $(*)$ becomes

$$({}^sf)(t) = f(s^{-1}t), \qquad s, t \in S. \tag{$**$}$$

The group $L \operatorname{wr}_S S$ has a useful faitful permutation representation on the set $L \times S$ given by

$${}^{fs}(x, t) = (f(st)x, st), \qquad f \in \textstyle\prod_S L,\ x \in L,\ s,\ t \in S. \tag{$3*$}$$

The embedding of $L \operatorname{wr}_\Omega S$ in $L \operatorname{wr}_S S$ then induces a faithful action of the former group on $L \times S$.

(3.1) *Generalized Wreath Products*

In the remainder of this section X denotes a group with a normal subgroup $S = \prod_{\alpha \in \Omega} L_\alpha$, which is a direct product of groups L_α isomorphic to L and permuted transitively according to an action of X on the index set Ω; thus

$${}^x(L_\alpha) = L_{{}^x\alpha}, \qquad x \in X,\ \alpha \in \Omega.$$

Moreover except in 3.5 we assume $X = D.S$ is the semidirect product of D and some subgroup S.

There are two useful ways to identify D with functions f on Ω. The simplest is to let $f(\alpha)$ take values in L_α, and identify f with $\prod_{\alpha \in \Omega} f(\alpha) \in D$. The action of S on D then translates into

$$({}^sf)(\alpha) = {}^s(f(s^{-1}\alpha)), \qquad s \in S,\ \alpha \in \Omega. \tag{$4*$}$$

A second point of view, used in our earlier discussion of the wreath product, is to use the isomorphism of L with L_α and let f take values in L; this gives

$$({}^s f)(\alpha) = {}^{\tau_s(\alpha)}(f(s^{-1}\alpha)), \qquad s \in S, \alpha \in \Omega \tag{5*}$$

for some automorphism $\tau_s(\alpha)$ of L. In the case of wreath product, $\tau_s(\alpha) = 1$ for each $s \in S$ and $\alpha \in \Omega$, as reflected in Eq. $(*)$, after suitable translation of notation. More generally we have an embedding of X in $(L.\ \mathrm{Aut}(L))\,\mathrm{wr}_\Omega\, S$ extending the natural inclusion $\prod_\Omega L \to \prod_\Omega (L.\ \mathrm{Aut}(L))$, and mapping $s \in S$ to $s\tau_s$, where $\tau_s: \Omega \to \mathrm{Aut}(L)$ is the function mapping α to $\tau_s(\alpha)$. Having pointed out this embedding, in the remainder of this section we adopt the second point of view only for wreath products. In both cases we use the convention (introduced earlier) that functions on Ω may be viewed as functions on S constant on the cosets of the stabilizer U in S of some point of Ω.

(3.2) *The Correspondence Going Down*

Again U is the stabilizer in S of a point of Ω, so we regard Ω as the coset space S/U. Let $D' = \prod_{\alpha \in \Omega - \{U\}} L_\alpha$, so that U acts on D' and $DU/D' \cong L.U$. Let $\pi: DU \to DU/D'$ be the natural map. If S' is a complement to D in X then $U' = S' \cap DU$ is a complement to D in DU, so $\pi(U')$ is a complement to $\pi(D) \cong L$ in $\pi(DU) \cong L.U$. Let $\mu(S')$ be the image of $\pi(U')$ in $L.U$. If S'' is a complement conjugate to S' in X then S'' is conjugate to S' under D, so $\mu(S'')$ is conjugate to $\mu(S')$ under L. We now establish the converse.

(3.3) *Let S' and S'' be complements to D in X, and suppose $\mu(S')$ is conjugate to $\mu(S'')$ in $L.U$. Then S' is conjugate to S'' in X. Indeed if $\mu(S') = \mu(S'')$ then S' is conjugate to S'' under the action of any subgroup D_0 of D of the form $\prod_\Omega L_0$ with $L_0 \leqslant L$ and $D_0 S' = D_0 S''$.*

Proof. Without loss we may take $\mu(S') = \mu(S'')$. Let $\varphi: X \to (L.\ \mathrm{Aut}(L))\,\mathrm{wr}_\Omega\, S$ be the embedding described in 3.1. By definition of φ, $\varphi(S)$ is a complement to $D^* = \prod_\Omega (L.\ \mathrm{Aut}(L))$ in $D^*.S$, and then as S' and S'' are complements to D in $D.S.$ and $\varphi(D) = \varphi(DS) \cap D^*$, $\varphi(S')$ and $\varphi(S'')$ are complements to D^* in D^*S. $\mu(S') = \mu(S'')$ so as $\varphi(D) = \varphi(DS) \cap D^*$, we have $\pi^*(\varphi(U')) = \pi^*(\varphi(U''))$ and then $\mu^*(\varphi(S'')) = \mu^*(\varphi(S''))$, using the obvious notation. Now if L_0 and D_0 are as in the statement of the lemma, and $\varphi(S')$ is conjugate to $\varphi(S'')$ under $\varphi(D_0)$, the certainly S' is conjugate to S'' under D_0. Hence, passing to D^*S and preforming a suitable change of notation, we may assume $X = L\,\mathrm{wr}_\Omega\, S$ is a wreath product.

For $s \in S$, $Ds \cap S'$ and $Ds \cap S''$ contain unique elements which we denote by s' and s'' and let $\beta_s = s's^{-1}$ and $\gamma_s = s''s^{-1}$ be the corresponding elements of D. Notice that $s' = \beta_s \gamma_s^{-1} s''$, so as $D_0 S' = D_0 S''$, $\beta_s \gamma_s^{-1} \in D_0$.

Regarding β_s and γ_s as functions from Ω into L, this is equivalent to the assertion that $\beta_s(t)\gamma_s(t)^{-1} \in L_0$ for each $t \in S$.

Since $\mu(S') = \mu(S'')$ we have $\pi(U') = \pi(U'')$, where $U' = S' \cap DU$ and $U'' = S'' \cap DU$. Thus for $u \in U$, $\pi(u') = \pi(u'')$, so $\beta_u \gamma_u^{-1} \in \ker(\pi) = D_U$; equivalently $\beta_u(U) = \gamma_u(U)$, or using our convention embedding X in $L \operatorname{wr}_S S$, this becomes $\beta_u(1) = \gamma_u(1)$. Now from $(**)$ we obtain ${}^s\beta_u(s) = {}^s\gamma_u(s)$ for all $s \in S$. As a consequence of the definition of β_s and γ_s we have $\beta_{su} = \beta_s {}^s\beta_u$ and $\gamma_{su} = \gamma_s {}^s\gamma_u$. Then recalling that β_{su} and γ_{su} are constant on the cosets of U, we have $\beta_{su}(su)\gamma_{su}(su)^{-1} = \beta_{su}(s)\gamma_{su}(s)^{-1} = \beta_s(s)\gamma_s(s)^{-1}$. Hence if we define $\delta \in \prod_S L$ by $\delta(s) = \beta_s(s)\gamma_s(s)^{-1}$ we have δ constant on the cosets of U, so $\delta \in \prod_\Omega L = D$. Indeed as $\beta_s(s)\gamma_s(s)^{-1} \in L_0$, even $\delta \in D_0$.

Define δ' and δ'' in $\prod_S L$ by $\delta'(s) = \beta_s(s)$ and $\delta''(s) = \gamma_s(s)$. To show δ conjugates S'' to S' it is enough to show δ' conjugates S to S' and δ'' conjugates S to S''. For this we use the faithful action of $L \operatorname{wr}_S S$ on $L \times S$. First note that by $(3*)$

$$ {}^{\delta' s}(x, 1) = (\delta'(s)x, s) = (\beta_s(s)x, s) = {}^{\beta_s s}(x, 1) = {}^{s'}(x, 1) $$

for all $s \in S$ and $x \in L$. So for $s \in S$

$$ \begin{aligned} {}^{\delta' s}(x, t) &= {}^{\delta' st}(x, 1) = {}^{(st)'}(x, 1) = {}^{s' t'}(x, 1) \\ &= {}^{s' \delta' t}(x, 1) = {}^{s' \delta'}(x, t). \end{aligned} $$

We have shown $\delta' s = s' \delta'$, so that δ' conjugates S to S' as desired. The same argument shows δ'' conjugates S to S'', completing the proof of the lemma.

(3.4) *The Correspondence Going Up*

Recall the discussion of 1-cohomology in Section 2. By the hypothesis of this section we have an action of S on D and an action of U on L via the identification of L with the factor L_α of D stabilized by U. So $\Gamma(S, D)$ and $\Gamma(U, L)$ are defined.

Let $\mathscr{Y}$ be a set of right coset representatives for U in S containing 1 as the representative of the coset U. Define $u: S \to U$ by

$$ u(s) = sy^{-1}, \qquad y \in \mathscr{Y},\ s \in S \cap yU. $$

For x, $s \in S$ and $\gamma \in \Gamma(U, L)$ define

$$ f_s(x) = (\gamma_{u(x)}^{-1} \gamma_{u(xs)})^x $$

and define

$$ \tilde{\gamma}_s(x) = f_s(x^{-1}). $$

Observe first that

$$u(vs) = vu(s) \qquad \text{for all} \quad s \in S,\ v \in U$$

and hence by the cocycle condition we have

$$\gamma_{u(vs)} = \gamma_v^v \gamma_{u(s)}.$$

Then for $v \in U$ and $x \in X$,

$$\begin{aligned} f_s(vx) &= (\gamma_{u(vx)}^{-1} \gamma_{u(vxs)})^{vx} = ((\gamma_v^v \gamma_{u(x)})^{-1} \gamma_v^v \gamma_{u(xs)})^{vx} \\ &= (\gamma_{u(x)}^{-1} \gamma_{u(xs)})^x = f_s(x). \end{aligned}$$

Hence for each $s \in S$, $f_s\colon x \to f_s(x)$ is a function from S into L constant on the right cosets of U, so $\tilde{\gamma}_s \in \prod_\Omega L \leqslant \prod_S L$. Thus $\tilde{\gamma}\colon s \mapsto \tilde{\gamma}_s$ is a function from S into $\prod_\Omega L = D$.

We show next that $\tilde{\gamma}$ satisfies the cocycle condition. For if $s, t, x \in S$, then by (4$*$) we have

$$\tilde{\gamma}_s(x)({}^s\tilde{\gamma}_t)(x) = \tilde{\gamma}_s(x)^s\,(\tilde{\gamma}_t(s^{-1}x))$$

and then applying the definitions of $\tilde{\gamma}$, f_s, and f_t, we get

$$\begin{aligned} \tilde{\gamma}_s(x)^s\,(\tilde{\gamma}_t(s^{-1}x)) &= f_s(x^{-1})^s\,(f_t(x^{-1}s)) \\ &= (\gamma_{u(x^{-1})}^{-1} \gamma_{u(x^{-1}s)})^{x^{-1}s}((\gamma_{u(x^{-1}s)}^{-1} \gamma_{u(x^{-1}st)})^{x^{-1}s}) \\ &= (\gamma_{u(x^{-1})}^{-1} \gamma_{u(x^{-1}st)})^{x^{-1}} = f_{st}(x^{-1}) = \gamma_{st}(x). \end{aligned}$$

So the cocycle condition holds and hence

$$\gamma \mapsto \tilde{\gamma} \text{ is a map from } \Gamma(U, L) \text{ into } \Gamma(S, D). \tag{6$*$}$$

Next $\tilde{\gamma}^*(S)$ consists of the elements $\tilde{\gamma}_s s$, $s \in S$, so $\mu(\tilde{\gamma}^*(S))$ consists of the elements $\tilde{\gamma}_v(1)v$, $v \in U$. As $\tilde{\gamma}_v(1) = f_v(1) = \gamma_{u(1)}^{-1}\gamma_{u(v)} = \gamma_v$, we conclude:

$$\mu(\tilde{\gamma}^*(S)) = \gamma^*(u). \tag{7$*$}$$

(3.5) *The Nonsplit Case*

Together with (3.3) the result (7$*$) establishes Theorem 2 below under the hypothesis that X is a semidirect product $D.S$. We now drop the assumption that D has a complement in X and show

$$D \text{ has a complement in } X \text{ if } \pi(D) \text{ has a complement in } \pi(N_X(L)). \tag{8$*$}$$

Suppose W/D' is a complement to D/D' in $N_G(L)/D'$, and consider the homomorphism $X \to \mathrm{Sym}(X/W)$, where the latter is the symmetric group on the set of right cosets of W in X. The kernel is contained in W and intersects D trivially, so without loss we may assume the kernel is 1 and the map is an inclusion. Note L acts regularly on the points in LW/W, but fixes all the rest. No point is fixed by D, and the points moved by L_α, L_β are disjoint for distinct α, β.

The normalizer N of D in $\mathrm{Sym}(X/W)$ can now be seen to have a very nice form. The normalizer of L induces on the support of L a group $\tilde{L}$ in which L is a regular normal subgroup, and we have

$$N = \Big(\prod_{\Omega} \tilde{L}_\alpha\Big)\, \mathrm{Sym}(\Omega) \cong \tilde{L} \,\mathrm{wr}_\Omega\, \mathrm{Sym}(\Omega).$$

Since L is regular in $\tilde{L}$, it has a complement, call it V. Then $(\prod_\Omega V_\alpha)$ $\mathrm{Sym}(\Omega) \cong V \,\mathrm{wr}_\Omega\, \mathrm{Sym}(\Omega)$ is a complement to D in N. It follows that D has a complement in X as well, and (8$*$) is proved.

With a little further study of N above we could even reprove (7$*$). However, we will need the cocycles again momentarily. Meanwhile observe that (3.3), (7$*$), and (8$*$) establish Theorem 2.

The following section is not needed elsewhere in this paper.

(3.6) *Stability*

Notice that a complement V to D in $N_X(L)$ is not necessarily contained in a complement T to D in X; we are only guaranteed $D'V = D'N_T(L)$. What conditions on V guarantee $V \subseteq T$, or, equivalently, $V = N_T(L)$ for some complement T to D in X?

PROPOSITION. *If V is a complement to D in $N_X(L)$, then $V \subseteq T$ for some complement T to D in X if and only if*

$$DV^x \cap V \text{ is } D\text{-conjugate to } V^x \cap DV \text{ for each } x \in X.$$

Proof. First suppose $V \subseteq T$. Note that the condition on x depends only on the D-coset to which it belongs. Hence we may take $x \in T$. But now both V and V^* are contained in T, and we have

$$DV^x \cap V = DV^x \cap T \cap V = V^x \cap V = V^x \cap T \cap DV = V^x \cap DV.$$

Next suppose the conjugacy condition is satisfied. We will use the method of (3.4), carefully choosing the set $\mathscr{Y}$ of right coset representatives for U in S required there. Let $\mathscr{Z}$ be a set of (U, U) double coset representatives in S, and for each $z \in \mathscr{Z}$, let $\mathscr{Y}_z$ be a set of right coset representatives for $U^z \cap U$ in U. Finally, put $\mathscr{Y} = \bigcup_{z \in \mathscr{Z}} z\mathscr{Y}_z$.

Let $\beta \in \Gamma(U, D)$ and $\gamma \in \Gamma(U, L)$ be cocycles corresponding to V. Thus $\beta_s(1) = \gamma_s$ for $s \in U$. Choose $d_z \in D$ with $(DV^z \cap V)^{d_z} = V^z \cap DV$. Let

$s \in U \cap {}^zU$ and $t = s^z$. Then $t \in U \leqslant DV$, so $(\beta_s s)^z \in V^z \cap DV$. Also $\beta_t t \in DV^z \cap V$, so $(\beta_t t)^{d_z} \in V^z \cap DV$. As $D(\beta_t t)^{d_z} = Dt = D(\beta_s s)^z$ we conclude $(\beta_s s)^z = (\beta_t t)^{d_z}$. Now as t acts on L^z we conclude $(\beta_t(z^{-1})t)^{d_z} = \beta_s^z(z^{-1})t = (\beta_s(1))^z \cdot t$ (by (4∗)) $= \gamma_s^z \cdot t$, which we record:

$$(\beta_t(z^{-1})t)^{d_z} = \gamma_s^z \cdot t \qquad \text{for each } s \in U \cap {}^zU \text{ and } t = s^z. \tag{9∗}$$

Recall the definition of $u(x)$ and $\tilde{\gamma}$ from (3.4). As $z \in \mathscr{Y}$, $u(z) = 1$, so $\gamma_{u(z)} = 1$. Also $zt = sz$ with $s \in U$, so $u(zt) = s$. Then $\tilde{\gamma}_t(z^{-1}) = (\gamma_{u(z)}^{-1}\gamma_{u(zt)})^z = \gamma_s^z$, so from (9∗) we have

$$(\beta_t(z^{-1})t)^{d_z} = \tilde{\gamma}_t(z^{-1})t \qquad \text{for each } t \in U \cap U^z. \tag{10∗}$$

Let

$$\Gamma = (L^z)^U, \qquad D'_\Gamma = \prod_{M \in \Gamma - \{L^z\}} M,$$

β'_u and $\tilde{\gamma}'_u$ the projections of β_u and γ_u on D_Γ, and β', $\tilde{\gamma}'$ the corresponding maps of U into D_Γ. Then $\beta', \tilde{\gamma}' \in \Gamma(U, D_\Gamma)$ and by (10∗), $(\beta')^*(U \cap U^z)^{d_z} = (\tilde{\gamma}')^*(U \cap U^z) \bmod D'_\Gamma$, so by (3.3) there exists $e_z \in D_\Gamma$ with $(\beta')^*(U)^{d_z} = (\tilde{\gamma}')^*(U)$. Let $e = \prod_{z \in \mathscr{Z}} e_z$. Then $V^e = \beta^*(U)^e = \tilde{\gamma}^*(U) = \tilde{\gamma}^*(S) \cap UD$, completing the proof.

We conclude this section by remarking that Theorem 2 may be regarded as a strong version for nonabelian 1-cohomology of Shapiro's lemma in abelian cohomology. Similarly the proposition above is a generalization of a (less well known) stability property of the cohomology of induced modules.

4. The Proof of Theorem 3

Theorem 3 was first obtained in collaboration with R. Guralnick using elementary group theoretic techniques. We give a cohomological proof here. We first recall some facts about cohomology; standard references are [2] and [3]. If U is a module for a group Q, then $H^0(Q, U)$ denotes the fixed points of Q on U, as usual.

Assume throughout this section that G is a finite group, p is a prime, K is a field of characteristic p, and V is an irreducible KG-module. Observe that as V is irreducible we have:

(4.1) *If Q is a normal subgroup of G with $[V, Q] \neq 0$, then $H^0(Q, V) = 0$. In particular if G is faithful on V and $1 \neq Q \lhd G$, then $V = [Q, V]$ so $H^0(Q, V) = 0$.*

We need the following lemma which can be derived from 4.1 and the Hochschild–Serre sequence [3, (10.6)] or [5, p. 126, Remarque]:

(4.2) *$H^1(G, V) \cong H^0(G/Q, H^1(Q, V))$ for each normal subgroup Q of G with $[Q, V] \neq 0$. In particular if $H^1(G, V) \neq 0$ then $H^1(Q, V) \neq 0$.*

Here we are using the fact that as $Q \trianglelefteq G$ there is a natural action of G on $H^1(Q, V)$ with Q in the kernel of this action. We also need the following result:

(4.3) *Let S and T be irreducible KA and KB-modules, respectively, Then*

$$H^1(A \times B, S \otimes T) \cong (H^1(A, S) \otimes H^0(B, T)) \oplus (H^0(A, S) \otimes H^1(B, T)).$$

Finally, we prove the following elementary result:

(4.4) *Let $X \trianglelefteq G$ with G/X solvable and $V = [V, X]$. Then $\dim(H^1(G, V)) \leqslant \dim(H^1(X, E))$ for each irreducible KX-submodule E of V.*

Choose a counterexample to 4.4 with G/X of minimal order, let $X \leqslant Y \trianglelefteq G$ with Y maximal in G, and let A be an irreducible KY-submodule of V. If $Y \neq X$ then by minimality of G/X, $\dim(H^1(G, V)) \leqslant \dim(H^1(Y, A) \leqslant \dim(H^1(X, E))$ for each irreducible KX-submodule E of A. As all irreducible submodules for X on V are isomorphic to some G-conjugate of E, the lemma holds, contrary to our choice of G as a counterexample. Thus X is maximal in G, so $G = \langle g, X \rangle$ for some $g \in G$.

Form the semidirect product XV and let $U = C_{\mathrm{Aut}(VX)}(V) \cap C(XV/V)$. We recall U is an elementary abelian p-group with $U/V \cong H^1(X, V)$, and that the action of G on V induces an action on U, and hence also on U/V, with the latter action equivalent to the natural action of G on $H^1(X, V)$. In particular as $V = \langle E^G \rangle$, $H^1(X, V) = U/V = \bar{U} = \langle \bar{D}^G \rangle$, where $D/E = C_{U/E}(X)$, and $\bar{D} \cong H^1(X, E)$. Choose n maximal subject to $V_0 = \langle E^{g^i}: 0 \leqslant i \leqslant n \rangle = \bigoplus_{i=0}^n E^{g^i}$. As X is irreducible on E and $X \trianglelefteq G$, $E^{g^i} \leqslant V_0$ for all i, so $V = V_0$ as $V = \langle E^G \rangle$. So $\bar{U} = \bigoplus_{i=0}^n \bar{D}^{g^i}$. $H^1(G, V) = H^0(\underline{g}, \bar{U})$, so we must show $\dim(C_{\bar{U}}(g)) \leqslant \dim(\bar{D})$. Assume otherwise and let $\bar{W} = \langle \bar{D}^{g^i}: 0 \leqslant i < n \rangle$. Then there is $0 \neq \bar{w} \in C_{\bar{W}}(g)$. Let $\bar{w}_i$ be the projection of $\bar{w}$ on $\bar{D}^{g^i}$. Then $\sum \bar{w}_i = \bar{w} = \bar{w}^g = \sum \bar{w}_i^g$ with $\bar{w}_i^g \in \bar{D}^{g^{i+1}}$, so $\bar{w}_i^g = \bar{w}_{i+1}$ for $0 \leqslant i < n-1$, and $\bar{w}_{n-1}^g = \bar{w}_0$. Therefore $\bar{w}_{n-1}^g = \bar{w}_0 \in \bar{D}^{g^n} \cap \bar{D} = 0$, so by induction on i, $\bar{w}_{i+1} = \bar{w}_i^g = 0$ for $0 \leqslant i < n-1$. But now $\bar{w} = 0$, a contradiction. Thus 4.4 is established.

Now to the proof of Theorem 3. Assume that G is faithful on V and $H^1(G, V) \neq 0$. If k is an extension of K then $H^1(G, V \otimes k) \cong H^1(G, V) \otimes k$, so replacing K by an algebraically closed extension, we may assume K is algebraically closed. By 4.1, $O_p(G) = 1$. $H^1(O_{p'}(G), V) = 0$ (by a Frattini argument for example) so by 4.2, $O_{p'}(G) = 1$. Hence $M = F^*(G)$ is the direct product of simple components. Let L be one of these components and $X = C_M(L)$. By Clifford's theorem, V is a semisimple KM-module whose homogeneous components V_i, $1 \leqslant i \leqslant n$, are permuted transitively by G. By 4.2, $H^1(M, V) \neq 0$, so $H^1(M, V_i) \neq 0$ for some i, and then by the transitive action of G, for all i. Similarly $H^1(M, W) \neq 0$ for each irreducible KM-submodule W of V.

As $M = L \times X$ and K is algebraically closed, $W \cong S \otimes T$ for irreducible KL and KX submodules S and T of V, respectively. Choose W so that $[L, W] \neq 0$. Then $S = [L, S]$, so $H^0(L, S) = 0$. Hence by (4.3), $0 \neq H^0(X, T)$, and then as T is a simple KX-module, $[T, X] = 0$. We have shown that $W = [L, W]$ and $[X, W] = 0$, so $V_i = [V_i, L]$ and $[X, V_i] = 0$ for any homogeneous component V_i with $[V_i, L] \neq 0$. As G is transitive on the homogeneous components of M, it follows that (1) and (2) hold. Now by Shapiro's lemma [ ] (or Theorem 2):

(4.5) $H^1(G, V) \cong H^1(N_G(L), U)$.

Let $Y = C_G(L)$, $A = N_G(L)$, and $B = YL$. As G is irreducible on V, A is irreducible on U. From (4.2) we get:

(4.6) $H^1(A, U) \cong H^0(A/B, H^1(B, U))$.

Then from 4.5 and 4.6 we conclude $H^1(B, U) \neq 0$. Next applying the argument of the last paragraph to the action of $B = Y \times L$ on U we obtain (3). After that we observe:

(4.7) $H^1(B, U) = H^1(L, U)$.

For if not there is a complement B' to U in BU with $Y \neq B' \cap YU = Y'$. By (3), $U \leqslant Z(YU)$, while $Y' \trianglelefteq B'$ so $Y' \trianglelefteq B'U = BU$. Thus as $[L, YU] = U$, $[L, Y'] \leqslant Y' \cap U = 0$, so $Y' = C_{YU}(L) = Y$, a contradiction. So 4.7 is established.

From 4.5–4.7 we get $H^1(G, V) \cong H^0(A/B, H^1(L, U)) \cong H^0(A/Y/B/Y, H^1(B/Y, U)) \cong H^0(A/Y, U)$, with the last isomorphism following from 4.2. Therefore (4) holds.

Finally, from the classification of the finite simple groups, the Schreier conjecture holds, so $\mathrm{Out}(L)$ is solvable. Hence (4) and 4.4 imply (5).

This completes the proof of Theorem 4.

The Schreier conjecture, and hence the classification of the finite simple groups, is used in the proof of part (5) of Theorem 3, and, via 6.3 and 6.5, to show the map in part (C)(3) of Theorem 1 is a surjection. These results are not needed elsewhere in the paper, which is otherwise independent of the classification.

5. More on Diagonal Subgroups

In this section we continue the hypothesis and notation of Section 2. The fixed points of the action of G on A will be denoted by $C_A(G)$ or $H^0(G,A)$. The sets $H^0(G,A)$ and $H^1(G,A)$ are "pointed" in that they have distinguished elements: The identity of the group $H^0(G,A)$ and the trivial class in $H^1(G,A)$.

If $\alpha: X \to Y$ and $\beta: Y \to Z$ are maps of pointed sets with distinguished points x_0, y_0, and z_0 in X, Y, and Z, respectively, then

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$$

is said to be *exact* if the image of X under α is the full preimage in Y of z_0 under β.

Suppose now G acts on a group B and A is a G-stable subgroup of B. Let C be the coset space $B/A = \{bA: b \in B\}$, and notice that $C_B(G)$ acts naturally by left multiplication on $H^0(G, C)$, with the latter regarded as a pointed set with distinguished point the coset A.

(5.1) There is a natural exact sequence of pointed sets

$$1 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \xrightarrow{\delta} H^1(G,A) \to H^1(G,B)$$

in which the map δ is given by $\delta(bA) = [\gamma]$ where $\gamma_g = b^{-1}\,{}^{g}b$, for $g \in G$, and the remaining maps are the obvious ones. Moreover the fibers of δ are the orbits of $C_B(G)$ in its natural action by left multiplication on $H^0(G, C)$.

Proof. Except for the last assertion and the fact that we have not assumed B normal in A, this is just Proposition 1 on page 133 of [5]. The details are straightforward with the possible exception of the transitivity of $C_B(G)$ on the fibers of δ. Suppose $\delta(bA) = \delta(dA)$ for some bA, $dA \in C$. Then there is $a \in A$ with $ab^{-1}({}^{g}b)\,{}^{g}a^{-1}) = d^{-1}({}^{g}d)$ for all $g \in G$, so that $x = dab^{-1} \in C_B(G)$. But of course $xbA = dA$.

Apparently the first person to write down this sequence without the normalizer assumption was Giraud in his book "*Cohomologie nonabeliene.*"

In the remainder of this section we assume:

Hypothesis 5.2. *G is a group containing a normal subgroup D which is the direct product of the G-conjugates $(L_i: i \in I)$ of some subgroup L. Let $\mathscr{D}$ be the set of D-classes of full diagonal subgroups of D (as defined in Section* 1) *and let $\mathscr{D}(G)$ consist of the classes in $\mathscr{D}$ stable under G-conjugation.*

As we saw in the introduction, $\mathscr{D}(G)$ is important in the study of maximal subgroups. Our aim here is to give a parameterization of $\mathscr{D}(G)$ when $\mathscr{D}(G)$ is nonempty; this is accomplished in Theorem 5 below. We do not address the question of whether $\mathscr{D}(G)$ is nonempty. Thus the situation is analgous to a familiar one in the theory of group extensions, where one asks first if the extension splits, and if the answer is positive one has at least a theoretical parameterization of the classes of complements by H^1, which may be computed explicitly in favorable circumstances.

To begin, let $\mathrm{Aut}_0(D)$ denote the subgroup of $\mathrm{Aut}(D)$ fixing L_i for each $i \in I$. Then $\mathrm{Aut}_0(D)$ is the direct product of the groups $\mathrm{Aut}(L_i)$, $i \in I$. Let $\mathrm{Out}_0(D)$ be the image of $\mathrm{Aut}_0(D)$ in $\mathrm{Out}(D)$ and assume $A_1^D \in \mathscr{D}(G)$. This implies $G = DN_G(A_1)$. By 1.2, $\mathrm{Aut}_0(D)$ is transitive on the full diagonal subgroups of D, and by 1.1 the stabilizer in $\mathrm{Aut}_0(D)$ of A_1 is a full diagonal subgroup isomorphic to $\mathrm{Aut}(A_1)$; we identify $\mathrm{Aut}(A_1)$ with this stabilizer. The action of $\mathrm{Aut}_0(D)$ induces a transitive action of $\mathrm{Out}_0(D)$ on $\mathscr{D}$, in which the stabilizer of A_1^D is $\mathrm{Out}(A_1)$, so we have a bijection of sets $\mathscr{D} \cong \mathrm{Out}_0(D)/\mathrm{Out}(A_1)$. We also have the conjugation map $G \to \mathrm{Aut}(D)$, whose image $\mathrm{Aut}_G(D)$ acts on $\mathrm{Aut}(A_1)$ since $G = N_G(A_1)D$, and also acts on the normal subgroup $\mathrm{Aut}_0(D)$ of $\mathrm{Aut}(D)$. This induces the map $G \to \mathrm{Out}(D)$ whose image $\mathrm{Out}_G(D)$ acts on $\mathrm{Out}(A_1)$ and $\mathrm{Out}_0(D)$. The permutation representation of G on $\mathscr{D}$ factors through the map $G \to \mathrm{Out}_G(D)$ and $\mathrm{Out}_G(D)\,\mathrm{Out}(A_1)$ is the stabilizer of A_1^D and the coset $\mathrm{Out}(A_1)$ in the transitive representations of $\mathrm{Out}_G(D)\,\mathrm{Out}_0(D)$ on $\mathscr{D}$ and $\mathrm{Out}_0(D)/\mathrm{Out}(A_1)$, respectively, so the isomorphism $\mathscr{D} \cong \mathrm{Out}_0(D)/\mathrm{Out}(A_1)$ is G-equivariant. We record these observations as:

(5.3) *There is an isomorphism of sets* $\mathscr{D} \cong \mathrm{Out}_0(D)/\mathrm{Out}(A_1)$ *defined by* $({}^{\beta}A_1)^D \to \bar{\beta}\,\mathrm{Out}(A_1)$, *where* $\beta \in \mathrm{Aut}_0(D)$ *and* $\bar{\beta}$ *is the image of* β *in* $\mathrm{Out}_0(D)$. *This isomorphism is G-equivariant.*

From 5.1 and 5.3 we obtain:

(5.4) *There is an exact sequence of pointed sets*

$$1 \to H^0(G, \mathrm{Out}(A_1)) \to H^0(G, \mathrm{Out}_0(D)) \to \mathscr{D}(G) \xrightarrow{\delta} H^1(G, \mathrm{Out}(A_1))$$
$$\to H^1(G, \mathrm{Out}_0(D))$$

with A_1^D *the distinguished point of* $\mathscr{D}(G)$ *and with* δ *and its fibers described in* 5.1.

We next observe that by 1.1.3:

(5.5) *G acts by inner automorphisms on* $\mathrm{Out}(A_1)$, *in the sense of* 2.2. *Indeed* $\mathrm{Out}_G(D)\,\mathrm{Out}(A_1) = \mathrm{Out}(A_1) \times Y(A_1)$, *where* $Y(A_1)$ *is the group of outer automorphisms induced on D by* $C_{\mathrm{Aut}(A_1)\mathrm{Aut}_G(D)}(A_1)$.

Hence we can apply 2.2 to conclude:

(5.6) *$H^1(G, \mathrm{Out}(A_1))$ is isomorphic to $\mathscr{H}\mathrm{om}(G, \mathrm{Out}(A_1))$ under the isomorphism of 2.2. This isomorphism together with the projection of $\mathrm{Out}(A_1)$ onto $\mathrm{Out}(L)$ induces an isomorphism $H^1(G, \mathrm{Out}(A_1)) \cong \mathscr{H}\mathrm{om}(G, \mathrm{Out}(L))$.*

Let $A^D \in \mathscr{D}(G)$ and define $\nu^A: G \to \mathrm{Out}(A)$ to be the composition of the map $G \to \mathrm{Out}_G(D)$ with the projection $\mathrm{Out}(A)\,\mathrm{Out}_G(D) \to \mathrm{Out}(A)$ supplied by 5.5. Further let $\nu_L^A: G \to \mathrm{Out}(L)$ be the composition of ν^A with the projection $\mathrm{Out}(A) \to \mathrm{Out}(L)$.

(5.7) *ν_L^A depends only on the class A^D, not on the representative A.*

Proof. $\mathrm{Out}(A) = \mathrm{Out}(A^x)$ for each $x \in D$ since $\mathrm{Aut}(A)\,\mathrm{Inn}(D) = \mathrm{Aut}(A^x)\,\mathrm{Inn}(D)$.

We can now state the main result of this section.

THEOREM 5. *Let G satisfy Hypothesis 5.2 and assume $\mathscr{D}(G)$ is nonempty. Let $\mathrm{Hom}_0(G, \mathrm{Out}(L))$ denote the set of homomorphisms in $\mathrm{Hom}(G, \mathrm{Out}(L))$ whose restriction to $N_G(L)$ is the conjugation map $N_G(L) \to \mathrm{Out}(L)$. Then the map*

$$\nu: \mathscr{D}(G) \to \mathrm{Hom}_0(G, \mathrm{Out}(L))$$

$$A^D \longmapsto \nu_L^A$$

is a bijection.

The proof of Theorem 5 involves a series of lemmas.

(5.8) $\nu_L^A \in \mathrm{Hom}_0(G, \mathrm{Out}(L))$.

Proof. Let $M = N_G(L)$. The conjugation map $M \to \mathrm{Out}(L)$ is the composition of $M \to \mathrm{Out}(D)$ with the projection $\pi_L: \mathrm{Out}_{N(L)}(D) \to \mathrm{Out}(L)$. On the other hand $\nu^A_{L\mid M}$ is the composition of $M \to \mathrm{Out}(D)$ with the projections $\pi_A: \mathrm{Out}_M(D)\,\mathrm{Out}(A) \to \mathrm{Out}(A)$ and π_L, so it suffices to show $\pi_L \circ \pi_A = \pi_L$. As $\mathrm{Aut}(A)\,\mathrm{Aut}_M(D)$ commutes with the projection $A \to L$, $\ker(\pi_A) \leqslant \ker(\pi_L)$. Also $\pi_L: \mathrm{Out}(A) \to \mathrm{Out}(L)$ is an isomorphism, so indeed $\pi_L \circ \pi_A = \pi_L$.

Tracing through the definitions and using 5.1, 5.3, 5.4, and 5.6, we conclude:

(5.9) (1) *$\delta(A^D) \in H^1(G, \mathrm{Out}(A_1))$ is the cohomology class of the cocycle $g \mapsto \bar{\beta}^{-1}({}^g\bar{\beta})$, where $\beta \in \mathrm{Aut}_0(D)$ with ${}^{\beta}A_1 = A$ and $\bar{\beta}$ denotes the image of β in $\mathrm{Out}_0(D)$.*

(2) *The image of $\delta(A^D)$ in $\mathscr{H}\mathrm{om}(G, \mathrm{Out}(A_1))$ under the isomorphism of 2.2 is the orbit of the homomorphism $g \mapsto \bar{\beta}^{-1}({}^g\bar{\beta})\, i(g)$, where $i = \nu^{A_1}: G \to \mathrm{Out}(A_1)$.*

(5.10) (1) *There exists* $\beta \in \mathrm{Aut}_0(D) \cap C(L)$ *with* ${}^{\beta}A_1 = A$.

(2) *If* β *is chosen as in part* (1), *then the composition of the homomorphism* $g \mapsto \bar{\beta}^{-1}({}^{g}\bar{\beta})\, i(g)$ *with the projection* $\mathrm{Out}(A_1) \to \mathrm{Out}(L)$ is just the map ν_L^A.

Proof. As $\mathrm{Aut}_0(D) = \mathrm{Aut}(A_1)(\mathrm{Aut}_0(D) \cap C(L))$ part (1) holds. Choose β as in part (1). $G = N_G(A_1)D$ and D is in the kernel of both the map $\pi\colon g \to \bar{\beta}^{-1}({}^{g}\bar{\beta})\, i(g)$ and ν_L^A, so it suffices to show ν_L^A and the composition of π with the projection $\mathrm{Out}(A_1) \to \mathrm{Out}(L)$ agree on $N_G(A_1)$.

Let $g \in N_G(A_1)$ and $\bar{\beta}^{-1}({}^{g}\bar{\beta}) = \alpha$, so that $\alpha \in \mathrm{Out}(A_1)$ and $\pi(g) = {}_{\alpha}i(g)$. Let ξ be the image of g in $\mathrm{Out}(D)$. Then ${}^{g}\bar{\beta} = {}^{\xi}\bar{\beta}$, so $\xi = \bar{\beta} \cdot \bar{\beta}^{-1}({}^{g}\bar{\beta})\, \xi\bar{\beta}^{-1} = \bar{\beta}\alpha\xi\bar{\beta}^{-1} = {}^{\bar{\beta}}(\alpha\xi)$. Now $\alpha\xi$ acts on $\mathrm{Out}(A_1)$ and ${}^{\beta}A_1 = A$, so $\xi = {}^{\bar{\beta}}(\alpha\xi)$ acts on $\mathrm{Out}(A)$. Thus $\xi = \xi_1\xi_2$ with $\xi_1 \in \mathrm{Out}(A)$ and $\xi_2 \in Y(A)$ by 5.5. Notice that $\nu_L^A(g)$ is the projection of ξ_1 on $\mathrm{Out}(L)$. Next $\xi^{\bar{\beta}} = \xi_1^{\bar{\beta}}\xi_2^{\bar{\beta}}$ with $\xi_1^{\bar{\beta}} \in \mathrm{Out}(A_1)$ and $\xi_2^{\bar{\beta}} \in Y(A_1)$, and as the projection of $\bar{\beta}$ on $\mathrm{Out}(L)$ is trivial, ξ_1 and $\xi_1^{\bar{\beta}}$ have the same projection $\nu_L^A(g)$ on $\mathrm{Out}(L)$. Finally, $\xi^{\bar{\beta}} = \alpha\xi$ with $\alpha \in \mathrm{Out}(A_1)$, so $\alpha i(g)$ is the projection of $\xi^{\bar{\beta}}$ on $\mathrm{Out}(A_1)$; that is, $\pi(g) = \alpha i(g) = \xi_1^{\bar{\beta}}$. Therefore $\nu_L^A(g) =$ projection of $\xi_1^{\bar{\beta}}$ on $\mathrm{Out}(L) =$ projection of $\pi(g)$ on $\mathrm{Out}(L)$, completing the proof.

(5.11) *Let* $\mathscr{H}\mathrm{om}_0(G, \mathrm{Out}(L))$ *denote the image of* $\mathrm{Hom}_0(G, \mathrm{Out}(L))$ *under the natural map assigning a homomorphism to its* $\mathrm{Out}(L)$*-conjugacy class. Then* $\mathscr{H}\mathrm{om}_0(G, \mathrm{Out}(L))$ *is the image of* $\delta(\mathscr{D}(G))$ *under the isomorphism* $H^1(G, \mathrm{Out}(A_1)) \cong \mathscr{H}\mathrm{om}(G, \mathrm{Out}(L))$ *of* 5.6.

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
H^1(G, \mathrm{Out}(A_1)) \arrow[r, "a"] \arrow[d, "b"'] & H^1(G, \mathrm{Out}_0(D)) \arrow[d, "c"] \\
H_1(N_G(L), \mathrm{Out}(A_1)) \arrow[r, "d"] & H^1(N_G(L), \mathrm{Out}(L))
\end{tikzcd}$$

The map a is induced by the inclusion $\mathrm{Out}(A_1) \to \mathrm{Out}_0(D)$, and by 5.4, $\delta(\mathscr{D}(G)) = a^{-1}(z_0)$, where z_0 is the distinguished point of $H^1(G, \mathrm{Out}_0(D))$. The map d is the isomorphism induced by the projection $\mathrm{Out}(A_1) \to \mathrm{Out}(L)$. The map b is the restriction map taking $[\gamma] \in H^1(G, \mathrm{Out}(A_1))$ to $[\gamma|_{N(L)}] \in H^1(N(L), \mathrm{Out}(A_1))$. Finally, the map c takes $[\gamma] \in H^1(G, \mathrm{Out}_0(D))$ to $[\pi \circ \gamma|_{N(L)}]$, where $\pi\colon \mathrm{Out}_0(D) \to \mathrm{Out}(L)$ is the projection. By Theorem 2, c is an isomorphism. As c and d are isomorphisms, $b^{-1}(x_0) = a^{-1}(z_0) = \delta(\mathscr{D}(G))$, where x_0 is the distinguished point of $H^1(N(L), \mathrm{Out}(A_1))$.

We have a second commutative diagram:

$$\begin{tikzcd} H^1(G, \mathrm{Out}(A_1)) \arrow[r, "e"] \arrow[d, "b"'] & \mathscr{H}\mathrm{om}(G, \mathrm{Out}(A_1)) \arrow[d, "B"] \\ H^1(N_G(L), \mathrm{Out}(A_1)) \arrow[r, "f"] & \mathscr{H}\mathrm{om}(N_G(L), \mathrm{Out}(A_1)) \end{tikzcd}$$

Here e is the isomorphism of 5.6, f is the restriction of e to $H^1(N_G(L), \mathrm{Out}(A_1))$, and B is defined by $B([\varphi]) = [\varphi|_{N(L)}]$. By definition of f in 2.2, $f(x_0)$ is the $\mathrm{Out}(A_1)$-orbit of ν^{A_1}. Hence

$$e(\delta(\mathscr{D}(G))) = B^{-1}(f(x_0)) = \{[\varphi] \in \mathscr{H}\mathrm{om}(G, \mathrm{Out}(A_1)) \colon \nu^{A_1} = \varphi|_{N(L)}\}$$

so that the lemma holds.

Let us review our progress to this point. We have the diagram:

$$\begin{tikzcd} \mathscr{D}(G) \arrow[r, "\delta"] \arrow[d, "\nu"'] & \delta(\mathscr{D}(G)) \arrow[d, "r"'] \\ \mathrm{Hom}_0(G, \mathrm{Out}(L)) \arrow[r, "s"] & \mathscr{H}\mathrm{om}_0(G, \mathrm{Out}(L)) \end{tikzcd}$$

where δ is the map of 5.3, $\nu \colon A^D \mapsto \nu_L^A$, r is the restriction of the map of 5.6 to $\delta(\mathscr{D}(G))$, and s is the natural map assigning a homomorphism to its $\mathrm{Out}(L)$-orbit. By definition δ and s are surjections, and by 5.11, r is an isomorphism. By 5.9 and 5.10, the diagram commutes.

To prove Theorem 5 we must show ν is an isomorphism, and from the last paragraph this is equivalent to the assertion that ν induces isomorphisms $\nu \colon \delta^{-1}(x) \mapsto s^{-1}(r(x))$ on the fibers of δ and s. The projection $C_{\mathrm{Out}_0(D)}(G) \to C_{\mathrm{Out}(L)}(N_G(L))$ defines an isomorphism which, by 5.1 and 5.4, induces a transitive action of $C_{\mathrm{Out}(L)}(N_G(L)) = M$ on the fibers of δ. By definition of s, M acts on the fibers of s. Thus to establish Theorem 5 it suffices to prove the following three lemmas:

(5.12) *M acts transitively on the fibers of s.*

(5.13) *ν commutes with the actions of M on the fibers of δ and s.*

(5.14) *$C_{\mathrm{Out}(L)}(\nu_L^A(G))$ is the stabilizer in M of both ν_L^A and A^D.*

We first prove 5.12. Suppose φ and ψ are in the same fiber of s. Then ${}^x\psi = \varphi$ for some $x \in \mathrm{Out}(L)$. As φ, $\psi \in \mathrm{Hom}_0(G, \mathrm{Out}(L))$, $\psi(g) = \varphi(g) = i(g)$, for $g \in N_G(L)$, where $i(g)$ is the image of g in $\mathrm{Out}(L)$ under the conjugation map. Thus ${}^{x}i(g)=i(g)$ for each $g\in N_G(L)$, so $x\in M$, establishing 5.12.

We next prove 5.14. First consider the stabilizer M_i in M of $i=v_L^A$. $x\in M$ fixes i precisely when $i(g)=i(g)^x$ for all $g\in G$. Thus $M_i=C_M(i(G))$. As $M=C_{\mathrm{Out}(L)}(i(N_G(L))$, $M_i=C_{\mathrm{Out}(L)}(i(G))$ as claimed. Next consider the stabilizer N of A^D in M. By 5.3 and the definition of the action of M on $\mathscr{D}$, N is the image under the projection map of the stabilizer N_0 in $C_{\mathrm{Out}_0(D)}(G)$ of the coset $\mathrm{Out}(A)\in\mathrm{Out}_0(D)/\mathrm{Out}(A)$. But $N_0=C_{\mathrm{Out}(A)}(G)=C_{\mathrm{Out}(A)}(v^A(G))$, so $N=C_{\mathrm{Out}(L)}(i(G))$, and 5.14 is established.

Finally, we prove 5.13. For $x\in\mathrm{Aut}_0(D)$ let $\bar{x}$ denote the image of x in $\mathrm{Out}_0(D)$. Let $\bar{\mathscr{E}}_L\in M$, $\mathscr{E}_L$ a preimage of $\bar{\mathscr{E}}_L$ in $\mathrm{Aut}(L)$, and $\mathscr{E}\in\mathrm{Aut}_0(D)$ such that $\bar{\mathscr{E}}_L$ is the projection of $\bar{\mathscr{E}}$ and $\bar{\mathscr{E}}$ is the preimage of $\bar{\mathscr{E}}_L$ under the projection isomorphism $C_{\mathrm{Out}_0(D)}(G)\to M$. Further let α be the preimage of $\mathscr{E}_L$ in $\mathrm{Aut}(A)$ under the projection isomorphism $\mathrm{Aut}(A)\to\mathrm{Aut}(L)$. Then $\beta=\mathscr{E}\alpha^{-1}\in\mathrm{Aut}_0(D)\cap C(L)$ with ${}^{\mathscr{E}}A={}^{\beta}A$, and by 5.10, $v(({}^{\mathscr{E}}A)^D)$ is the composition of the map $g\mapsto\bar{\beta}^{-1}({}^{g}\bar{\beta})\,i(g)$ with the projection isomorphism $p\colon\mathrm{Out}(A)\to\mathrm{Out}(L)$, where $i=v^A$. Now $\bar{\beta}^{-1}({}^{g}\bar{\beta})\,i(g)=\bar{\alpha}\bar{\mathscr{E}}^{-1}({}^{g}\bar{\mathscr{E}})({}^{g}\bar{\alpha}^{-1})\,i(g)=\bar{\alpha}({}^{g}\bar{\alpha}^{-1})\,i(g)$, as $\bar{\mathscr{E}}\in C_{\mathrm{Out}_0(D)}(G)$. Moreover $\bar{\alpha}\in\mathrm{Out}(A)$, so $({}^{g}\bar{\alpha})=i(g)\,\bar{\alpha}i(g)^{-1}$, and thus $\bar{\alpha}({}^{g}\bar{\alpha}^{-1})\,i(g)=\bar{\alpha}_{i(g)}$. Hence $v(({}^{\mathscr{E}}A)^D)=p\circ{}^{\bar{\alpha}}i={}^{(\bar{\mathscr{E}}_L)}(v_L^A)$. However, $(\mathscr{E}_A)^D$ is the image of A^D under $\bar{\mathscr{E}}_L\in M$, so 5.13 is established.

This completes the proof of Theorem 5.

We close this section with a corollary of Theorem 5.

(5.15) *Assume Hypothesis* 5.2 *and assume also that* $G=\langle N_G(L),x\rangle$ *for some* $x\in G$. *Then* $|\mathscr{D}(G)|\leqslant|\mathrm{Out}(L)|$.

Proof. Each $\varphi\in\mathrm{Hom}_0(G,\mathrm{Out}(L))$ is determined by $\varphi(x)$, so Theorem 5 gives the indicated bound.

6. The Proof of Theorem 1

In this section G is a finite group and $M\in\mathscr{M}$ with $\ker_M(G)=1$; that is $M\in\mathscr{M}^*$. Let $O_\infty(G)$ denote the largest solvable normal subgroup of G. If $O_\infty(G)\neq 1$ a well-known elementary argument shows that $F^*(G)$ is an elementary abelian p-group, M is a complement to $F^*(G)$ in G, and M acts irreducibly on $F^*(G)$ via conjugation. Indeed $\mathscr{C}^*\cong H^1(M,F^*(G))$, and hence part (A) of Theorem 1 holds by Theorem 3 (established in Section 4).

So we may assume throughout the remainder of this section that $O_\infty(G)=1$. Hence $F^*(G)$ is the direct product of the components of G, and each component is simple. Let L be a component of G, $\Delta=L^G$, and $D=\langle\Delta\rangle$. This notation differs slightly from that in the statement of

Theorem 1 in the introduction where $F^*(G)=D$, but in the next lemma we deal with case (B), hence quickly reducing to the case $F^*(G)=D$. As in the introduction, $\mathscr{C}$ denotes the conjugacy class of maximal subgroups of G and $\mathscr{C}^*$ the set of classes with a representative in $\mathscr{n}^*$. We also use the other notation and terminology established in the introduction.

(6.1) *Assume* $F^*(G)\neq D$. *Then*

(1) *There exists a component* K *of* G *and an isomorphism* $\alpha: K\to L$ *such that* $K\notin \Delta$, $N_G(L)=N_G(K)$, $F^*(G)=\langle (LK)^G\rangle$, $C_G(D)=\langle K^G\rangle\cong D$, $LK\cap M=\operatorname{diag}(\alpha)$, *and* $F^*(G)\cap M$ *is the direct product of the M-conjugates of* $\operatorname{diag}(\alpha)$.

(2) *Let* $\mathscr{U}(L)$ *consist of those* U *in* K^G *with* $N_G(L)=N_G(U)$ *and* $\mathscr{D}_{LU}(N_G(L))\neq\varnothing$, *and let* $\mathscr{S}$ *denote* $\coprod_{U\in\mathscr{U}(L)}\mathscr{D}_{LU}(N_G(UL))$. *Then for* $N\in\mathscr{n}^*$ *there exists* $U(N)\in\mathscr{U}(L)$ *such that the map* $\varphi: N^G\mapsto (N\cap LU(N))^{LU(N)}$ *is a bijection between* $\mathscr{C}^*$ *and* $\mathscr{S}$.

(3) $\mathscr{C}^*\simeq\coprod_{U\in\mathscr{U}(L)} C_{\mathrm{Out}(L)}(N_G(L))$.

Proof. Let $1\neq A\trianglelefteq G$ with $A\leqslant C_G(D)$. Set $B=AD\cap M$ and $A_M=\langle (A\cap M)^A\rangle$. $AD=AD\cap G=AD\cap MD=BD$. Then $A_M=\langle (A\cap M)^A\rangle=\langle (A\cap M)^{AD}\rangle=\langle (A\cap M)^{BD}\rangle=\langle (A\cap M)^B\rangle\leqslant M$. Also A_M is invariant under M and D, so $A_M\trianglelefteq MD=G$. Hence as $\ker_M(G)=1$, $A_M=1$, so $A\cap M=1$. Therefore $C_M(D)=1$.

We can interchange the roles of A and D in the argument of the last paragraph to show $AD=AB$ and $D\cap M=1$. Then $A\cong AD/D=BD/D\cong B$ and similarly $D\cong B$. As this holds both for $A=C_G(D)$ and for a minimal normal subgroup A of G contained in $C_G(D)$, we conclude $A=C_G(D)$ is a minimal normal subgroup of G isomorphic to D. In particular $F^*(G)=AD$.

We have also shown that $B=AD\cap M=\operatorname{diag}(\beta)$ is a full diagonal subgroup of AD, where $\beta: D\to A$ is the composition of the isomorphisms $\pi_D^{-1}: D\to B$ and $\pi_A: B\to A$, and π_D and π_A are the projections of B on D and A, respectively. As these porjections commute with the action of M by conjugation, so does β. Thus setting $K=L_\beta$, $N_M(K)=N_M(L)$, so $N_G(K)=N_G(L)$. Let $\alpha=\beta|_L$. Then $\operatorname{diag}(\alpha)=B\cap LK=M\cap LK$, and (1) is established.

Next $D\leqslant N_G(LK)$, so $N_G(LK)=DN_M(LK)$ and hence $(M\cap LK)^{LK}=(M\cap LK)^{DK}$ is invariant under $N_G(LK)$. Adopt the notation of 6.1.2. Then $K=U(M)\in\mathscr{U}(L)$ and φ takes $\mathscr{C}^*$ into $\mathscr{S}$.

Next let $K\in\mathscr{U}(L)$, $X^{LK}\in\mathscr{S}$, and Ω the set of subgroups $\prod_{W\in(LK)^G}X_W$, $X_W\in X^G\cap W$. By (1.3, D is transitive on Ω so by a Frattini argument, $G=DN_G(Y)$, for $Y\in\Omega$. By 1.1 and 1.2, $X=N_{LK}(X)$, so $Y=N_D(Y)$. As $G=DN_G(Y)$, $N_G(Y)$ is maximal in G by 1.5. Hence φ is a surjection. Suppose $X=LK\cap M$. Then by (1), $M\cap D\in\Omega$, so $M=N_G(M\cap D)\in N_G(Y)^G$. Thus φ is an injection and (2) is established.

Part (3) follows from 2.4 and (2).

Notice that Lemma 6.1 implies that the conclusions of part (B) of Theorem 1 hold when $F^*(G) \neq \langle L^G \rangle$. Thus in the remainder of this section we assume that $F^*(G) = \langle L^G \rangle = \langle \Delta \rangle = D$.

(6.2) *Either*

(1) $\mathrm{Aut}_M(L) = \mathrm{Aut}_G(L)$ *and* $L \cap M = 1$, *or*

(2) $\mathrm{Aut}_M(L)$ *is a maximal subgroup of* $\mathrm{Aut}_G(L)$, $\mathrm{Aut}_M(L) \cap L = M \cap L \neq L$, *and* $M \cap D$ *is the direct product of the M-conjugates of* $M \cap L$.

Proof. Let $X = N_G(L)$ and $X^* = X/C_G(L) = \mathrm{Aut}_G(L)$. $D \leqslant N_G(L)$, so $N_G(L) = N_M(L)D$ and then $X^* = L^*N_M(L)^*$.

Suppose $L^* \leqslant N_M(L)^*$. Then $X^* = N_M(L)^*$; that is, $\mathrm{Aut}_G(L) = \mathrm{Aut}_M(L)$. So $L \leqslant N_M(L)\, C_G(L)$. Now if $1 \neq L \cap M$ then $L = \langle (L \cap M)^L \rangle = \langle (L \cap M)^{LC_G(L)} \rangle \leqslant \langle (L \cap M)^{N_M(L)C_G(L)} \rangle = \langle (L \cap M)^{N_M(L)} \rangle \leqslant M$, contradicting $\ker_M(G) = 1$. Thus $L \cap M = 1$ and (1) holds.

So assume $L^* \nleqslant N_M(L)^*$. Let $N^* \in \mathscr{n}_{X^*}$ contain $N_M(L)^*$. As $X^* = L^*N_M(L)^*$ and $N_M(L)^* \leqslant N^*$, we have $N^* = (L \cap N)^* N_M(L)^*$. Let $B = \langle (L \cap N)^M \rangle$. As $L \cap N \trianglelefteq N_M(L)$, B is the direct product of the M-conjugates of $L \cap N$. Then $MB \leqslant G$ with $BN_M(L) = N_{MB}(L)$, so $N_{MB}(L)^* = N^*$. Hence MB is a proper subgroup of G, so as M is maximal, $B \leqslant MB = M$. If $d \in M \cap D - B$ then as M is transitive on Δ and $B \trianglelefteq M$, we may choose d so that $d^* \notin B^* = (L \cap N)^*$. Then as $N^* \in \mathscr{n}_{X^*}$, $X^* = \langle d^*, N^* \rangle \leqslant N_M(L)^*$, contrary to our assumption. Thus $M \cap D = B$ and (2) holds.

We now partition $\mathscr{n}^*$ into three subsets according to the three subcases of case (C) of Theorem 1:

$$\mathscr{n}_1^* = \{N \in \mathscr{n}^* \colon \mathrm{Aut}_N(L) = \mathrm{Aut}_G(L) \text{ and } N \cap D = 1\}$$

$$\mathscr{n}_2^* = \{N \in \mathscr{n}^* \colon \mathrm{Aut}_N(L) = \mathrm{Aut}_G(L) \text{ and } N \cap D \neq 1\}$$

$$\mathscr{n}_3^* = \{N \in \mathscr{n}^* \colon \mathrm{Aut}_N(L) \text{ is maximal in } \mathrm{Aut}_G(L)\}.$$

Let $\mathscr{C}_i^*$ denote those classes in $\mathscr{C}^*$ with a representative in $\mathscr{n}_i^*$. We record a lemma restricting members of $\mathscr{n}_3^*$.

(6.3) *Assume* $D = L$. *Then* $L \cap M \neq 1$ *and* $M = N_G(L \cap M)$.

Proof. $L \cap M \trianglelefteq M$, so if $L \cap M \neq 1$ then $M = N_G(L \cap M)$ by maximality of M. So assume $L \cap M = 1$. Then G is the semidirect product of L with M. Then M is isomorphic to a subgroup of $\mathrm{Out}(L)$, so M is solvable. Let X be a minimal normal subgroup of M. Then X is a p-group for some prime p. By maximality of M, $M = N_G(X)$. So $C_L(X) = 1$. Thus L is a p'-

group and there exists a unique X-invariant Sylow 2-group T of L. Then $M = N_G(X) \leqslant N(T)$, so $M \leqslant MT < G$, a contradiction.

Let $\pi: N_G(L) \to N_G(L)/C_D(L)$ be the natural map, and define $\mathscr{M}_1'$ and $\mathscr{C}_1'$ as in Theorem 1; that, $\mathscr{M}_1'$ is the set of complements N to D in G with $\mathrm{Aut}_N(L) = \mathrm{Aut}_G(L)$, and $\mathscr{C}_1'$ is the set of orbits of G on $\mathscr{M}_1'$. Observe that by Theorem 2, the map

$$\varphi: X^G \mapsto (N_X(L)\pi)^{L\pi}$$

is a bijection between $\mathscr{C}_1'$ and the set of $L\pi$-classes $(Y\pi)^{L\pi}$ of complements $Y\pi$ to $L\pi$ in $N_G(L)\pi$ such that $\mathrm{Inn}(L\pi) \leqslant \mathrm{Aut}_{Y\pi}(L\pi)$. By definition $\mathscr{M}_1^* \subseteq \mathscr{M}_1'$, and conversely a member X of $\mathscr{M}_1'$ is contained in $\mathscr{M}_1^*$ if and only if X is contained in no member of $\mathscr{M}_2^*$. Thus the conclusion of the first paragraph of part (C)(1) of Theorem 1 holds. We postpone the discussion of the second paragraph until the next section.

The next lemma handles $\mathscr{C}_2^*$.

(6.4) (1) *If* $M \in {}_2^*$ *then there exists* $\Gamma \in \mathscr{P}^*(G, L)$ *such that* $M \cap D_\Gamma \in \mathscr{F}_\Gamma$ *and* $M \cap D$ *is the direct product of the* M*-conjugates of* $M \cap D_\Gamma$.

(2) *The map* $M^G \mapsto (M \cap D_\Gamma)^{D_\Gamma}$ *is a bijection of* $\mathscr{C}_2^*$ *with*

$$\coprod_{\Gamma \in \mathscr{P}^*(G,L)} \mathscr{D}_{D_\Gamma}(N_G(D_\Gamma)).$$

(3) $|\mathscr{C}_2^*| \leqslant |\mathscr{P}^*(G, L)|\,|\mathrm{Out}(L)|$.

Proof. If $M \in \mathscr{M}_2^*$ then $1 \neq M \cap D$, so $M \cap D$ projects nontrivially on some member of Δ. As $M \cap D \trianglelefteq M$ and M is transitive on Δ, $M \cap D$ projects nontrivially on L. Then $1 \neq \mathrm{Aut}_{M \cap D}(L) \trianglelefteq \mathrm{Aut}_G(L)$, so as $L = F^*(\mathrm{Aut}_G(L))$, the projection of $M \cap D$ on L is surjective. So 1.5 implies part (1). Indeed 1.5, 1.3, and a Frattini argument yield part (2).

To prove (3) we may assume $\mathscr{S} = \mathscr{D}_{D_\Gamma}(N_G(D_\Gamma))$ is nonempty for some $\Gamma \in \mathscr{P}^*(G, L)$, and it remains to show $|\mathscr{S}| \leqslant |\mathrm{Out}(L)|$. But this follows from 5.15, since by minimality of Γ^G, $N_G(D_\Gamma)$ is primitive on Γ, so $N_G(L)$ is maximal in $N_G(D_\Gamma)$.

Finally, to complete the proof of Theorem 1, we deal with $\mathscr{C}_3^*$.

(6.5) *The map* $N^G \to \mathrm{Aut}_N(L)^{\mathrm{Inn}(L)}$ *is a bijection of* $\mathscr{C}_3^*$ *with the set of conjugacy classes of maximal subgroups of* $\mathrm{Aut}_G(L)$ *not containing* $\mathrm{Inn}(L)$.

Proof. Let $X = N_G(L)$ and $X^* = X/C_G(L) = \mathrm{Aut}_G(L)$. By 6.2, $N_M(L)^*$ is maximal in X^* if $M \in \mathscr{M}_3$, and $M \cap D$ is the direct product of the M-conjugates of $M \cap L$, so the map

$$\varphi: N^G \mapsto (N \cap X)^{*X^*}$$

takes $\mathscr{C}_3^*$ into the set $\mathscr{S}$ of conjugacy classes of maximal subgroups of X^* which do not contain L^*. If $Y^{*X^*} \in \mathscr{S}$ then $L \cap Y \neq 1$, $X = LN_X(L \cap Y)$, and $N_X(L \cap Y)$ is maximal in X by 6.3. Let Ω be the set of subgroups $\prod_{K \in \Delta} W_K$, where $W_K \in (L \cap Y)^G \in K$, and let $W \in \Omega$. By 1.3 and a Frattini argument, $G = DN_G(W)$. A second application of 1.3 shows $\Gamma = \{ \in \Omega : Z \cap L\}$ is invariant under $V = C_G(L)$ and $C_D(L)$ is transitive on Γ, so choosing $W \cap L = Y \cap L$, we have $V = C_D(L)\, N_V(W)$. Thus $N_X(L \cap Y)^* = N_X(W)^*$. Hence $N_G(W)^G \in \mathscr{C}_3^*$ is mapped onto Y^{*X^*} under φ, so φ is a surjection. By 1.3 and 6.2, φ is an injection.

Notice we have completed the proof of Theorem 1, modulo the proof of Theorem 4. Theorem 4 is established in the next section.

7. The Proof of Theorem 4

In this section we assume the hypothesis and notation of Theorem 4. We investigate which members of $\mathscr{M}_1'$ are $\mathscr{M}_2^*$. In general $\mathscr{M}_1^*$ is properly contained in $\mathscr{M}_1'$; for example, the argument in 6.3 shows:

(7.1) *If $O_\infty(G/D) \neq 1$ then $\mathscr{C}_1^*$ is empty.*

Recall $O_\infty(H)$ is the largest normal solvable subgroup of a group H. If $G = A_5 \operatorname{wr}_\Omega S$ where S is a split extension of E_{16} by A_5 and Ω is of order 16, then by 7.1, $\mathscr{C}_1^*$ is empty, while by Theorem 2 $\mathscr{M}_1'$ is nonempty. On the other hand there are many examples of groups G with $\mathscr{C}_1^*$ nonempty, as the next lemma shows.

(7.2) *If no subgroup of G/D properly containing $N_G(L)/D$ has a homomorphic image whose generalized Fitting subgroup is isomorphic to L, then $\mathscr{C}_1^* = \mathscr{C}_1'$.*

Proof. Suppose $T \in \mathscr{M}_1'$ and $T < X \in \mathscr{M}$. Then $X = T(X \cap D)$ with $X \cap D \neq 1$, so $X \in \mathscr{M}_2^*$. By Theorem 1 there is $\Gamma \in \mathscr{P}^*(G, L)$ and $A \in \mathscr{F}_\Gamma$ such that $A = X \cap D_\Gamma$. Then $U = N_T(L) < V = N_T(A)$ and $\operatorname{Inn}(L) \leqslant \operatorname{Aut}_U(L) = \operatorname{Aut}_U(A) \leqslant \operatorname{Aut}_V(A)$, so $F^*(V/C_V(A)) \cong L$, contrary to the hypothesis of this lemma.

Observe that if $G \cong A_n \operatorname{wr}_\Omega A_{n+1}$, where Ω is of order $n+1$, then by 7.2, $\mathscr{C}_1^*$ is nonempty.

(7.3) *Let L be normal in a group H, $K = C_H(L)$, Γ the set of all normal subgroups J of H contained in K with $H/JL \cong H/K$, $I = \bigcap_{J \in \Gamma} J$, and $H^* = H/I$. Let $\mathscr{A}$ be the set of conjugacy classes X^H of complements X to L in H with $H = XK$. Then*

(1) *K^* is the direct product of components K_i^*, $1 \leqslant i \leqslant n$, isomorphic to L and normal in H^*. Moreover $\Gamma = \{H_i : 1 \leqslant i \leqslant n\}$, where $H_i^* = \prod_{j \neq i} K_j^*$.*

(2) *Let $\mathscr{B} = \bigcup_{i=1}^n \mathscr{D}_{(LK_i)^*}(H^*)$. Then for $X^H \in \mathscr{A}$, $(X \cap LK)^* = H_i^* \times (X \cap LK_i)^*$ for some i, and the map $X^H \to (X \cap LK_i)^{*LK}$ is a bijection of $\mathscr{A}$ with $\mathscr{B}$.*

(3) *$\mathscr{A}_i = \{X^H : H_i = X \cap K\}$ is of order 0 or $|C_{\mathrm{Out}(L)}(\mathrm{Out}_H(L))|$, so $|\mathscr{A}| = m\,|C_{\mathrm{Out}(L)}(\mathrm{Out}_H(L))|$ where m is the number of i, $1 \leqslant i \leqslant n$, with $\mathscr{D}_{(LK_i)^*}(H^*)$ nonempty.*

Proof. Let $\Omega \subseteq \Gamma$ be of maximal order k subject to $\tilde{K} = K/I_\Omega = \tilde{K}_1 \times \cdots \times \tilde{K}_k$, with $\tilde{K}_i \cong L$, $I_\Omega = \bigcap_{J \in \Omega} J$, and $\Omega = \{H_i : 1 \leqslant i \leqslant k\}$, where $\tilde{H}_i = \prod_{j \neq i} \tilde{K}_j$. Let $J \in \Gamma$. $\tilde{J} \trianglelefteq \tilde{K}$ with $K/J \cong L$, so either $\tilde{J} = \tilde{K}$ or $\tilde{K}/\tilde{J} \cong K/J$ and hence $J = H_i \in \Omega$; we may assume the former. Let $\theta = \{J\} \cup \Omega$ and $K\alpha = K/I_\theta$. Then $K\alpha = J\alpha \times (I_\Omega)\alpha$ with $J\alpha \cong \tilde{K}$ and $(I_\Omega)\alpha \cong K/J \cong L$. But now the maximality of k is violated.

So (1) holds. Let $X^H \in \mathscr{A}$. Then $X \cap KL$ is a complement to L in KL, so $X \cap K \trianglelefteq LX = H$, and as $H = XK$, $(X \cap KL)/(X \cap K)$ is also a complement to $K/(X \cap K)$ in $KL/(X \cap K)$. Thus $LK/(X \cap K) = L(X \cap K)/(X \cap K) \times K/(X \cap K) \cong L \times L$, so that part (1) and 6.1 imply parts (2) and (3).

(7.4) *Assume $\Gamma \in \mathscr{P}^*(G, L)$ and $A^{D_\Gamma} \in \mathscr{D}_{D_\Gamma}(N_G(D_\Gamma))$. Then*

(1) *$DC_G(A)$ is normal in $N_G(D_\Gamma)$ and is the kernel of the map $N_G(D_\Gamma) = DN_G(A) \to N_G(A)/N_D(A) \to \mathrm{Out}(A)$.*

(2) *$N_G(L) \cap DC_G(A) = DC_G(L)$.*

(3) *Either $N_G(L)\,DC_G(A) = N_G(D_\Gamma)$ or $DC_G(A) = DC_G(L)$.*

(4) *$DC_M(A) \trianglelefteq N_G(D_\Gamma)$ if $G = MD$.*

Proof. If $G = MD$ then $DC_M(A) \trianglelefteq DN_M(A) = DN_M(D_\Gamma) = N_G(D_\Gamma)$, so (4) holds. (1) is trivial once we observe $\mathrm{Aut}_{N_D(A)}(A) = \mathrm{Inn}(A)$.

$D \leqslant N_G(L)$, so $N_G(L) \cap DC_G(A) = D(N_G(L) \cap C(A))$. As $\Gamma \in \mathscr{P}^*(G, L)$, $N_G(L) \leqslant N_G(D_\Gamma)$. As A^{D_Γ} is $N_G(D_\Gamma)$-invariant, $DC_G(L) = D(N(A) \cap DC(L))$. The projection map $A \to L$ is $N_G(A) \cap N_G(L)$-equivariant, so $N_G(L) \cap C(A) \leqslant C_G(L)$, and as $DC(L)$ induces inner automorphisms on L, $N(A) \cap DC(L)$ induces inner automorphisms on A so $N(A) \cap DC(L) \leqslant N(L) \cap AC_G(A) \leqslant N(L) \cap DC_G(A)$. Therefore (2) holds.

As $\Gamma \in \mathscr{P}^*(G, L)$, $N_G(L)$ is maximal in $N_G(D_\Gamma)$. Thus as $DC_G(A) \trianglelefteq N_G(D_\Gamma)$ by (1), either $N_G(D_\Gamma) = N_G(L)\,DC_G(A)$ or $DC_G(A) \leqslant N_G(L)$, and since it remains only to prove (3), we may assume the latter. Then $DC_G(A) = DC_G(L)$ by (2), so (3) holds.

For $\Gamma \in \mathscr{P}^*(G, L)$ let $\mathscr{n}_{2,\Gamma}^*$ consist of those $N \in \mathscr{n}_2^*$ with $N \cap D_\Gamma \in \mathscr{F}_\Gamma$. By 6.4, $\mathscr{n}_2^*$ is partitioned by $(\mathscr{n}_{2,\Gamma}^* : \Gamma \in \mathscr{P}^*(G, L))$.

THEOREM 7.5. *Let $M \in \mathscr{N}_1'$ and set $E = DC_M(L)$. Let $\Gamma \in \mathscr{P}^*(G, L)$ with $\mathscr{N}_{2,\Gamma}^*$ nonempty. Then M is contained in a member of $\mathscr{N}_{2,\Gamma}^*$ if and only if either*

(a) *E and $DC_G(L)$ are normal in $N_G(D_\Gamma)$, or*

(b) *the section $N_G(L)/E$ has a normal complement in $N_G(D_\Gamma)$.*

The proof of Theorem 7.5 involves a series of lemmas.

(7.6) *If M is contained in a member of $\mathscr{N}_{2,\Gamma}^*$ then* (a) *or* (b) *holds.*

Proof. Assume $N \in \mathscr{N}_{2,\Gamma}^*$ with $M \leqslant N$, and set $A = N \cap D_\Gamma$. Then $N_M(L) \leqslant N_G(A)$ so that the projection $A \to L$ is $N_M(L)$-equivariant. Hence $C_M(L) = C_M(A) \cap N_G(L)$ and therefore

(7.6.1) $DC_M(A) \cap N_G(L) = DC_M(L) = E.$

Let $x \in M \cap DC_G(A)$. As $M \in \mathscr{N}_1'$, $\mathrm{Inn}(L) \leqslant \mathrm{Aut}_M(L)$ and then from the $N_M(L)$-equivariance also $\mathrm{Inn}(A) \leqslant \mathrm{Aut}_{N_M(L)}(A)$. So there exists $y \in N_M(L)$ with $y^{-1}x \in C_M(A)$. Thus $M \cap DC_G(A) \leqslant N_M(L)\, C_M(A)$, do

(7.6.2) $DC_G(A) = D(M \cap DC_G(A)) \leqslant N_M(L)\, DC_M(A).$

By 7.5.3 either $DC_G(A) = DC_G(L)$ or $N_G(L)\, DC_G(A) = N_G(D_\Gamma)$. In the first case $DC_G(L) \trianglelefteq N_G(D_\Gamma)$ by 7.4.1, and also $DC_M(A) \leqslant DC_G(L) \leqslant N_G(L)$, so by 7.6.1, $DC_M(A) = DC_M(L)$. So by 7.4.4, $DC_M(L) \trianglelefteq N_G(D_\Gamma)$. Thus (a) holds.

So assume the second case holds. Then by 7.4.4 and 7.6.2, $DC_M(A) \trianglelefteq N_D(D_\Gamma) = N_G(L)\, DC_M(A)$, so by 7.6.1 $DC_M(A)$ is a normal complement to $N_G(L)/E$ in $N_G(D_\Gamma)$. Thus (b) holds, completing the proof of the lemma.

Because of 7.6 we may assume during the remainder of the proof of Theorem 7.5 that (a) or (b) holds; to begin assume (b) holds. Let K be the normal complement guaranteed by (b). Then $N_G(D_\Gamma) = KN_G(L)$ and $K \cap N_G(L) = DC_M(L)$, so the map

(7.7) $N_G(D_\Gamma) \to KN_G(L)/K \to \mathrm{Out}_G(L) \to \mathrm{Out}(L)$

extends the conjugation map $N_G(L) \to \mathrm{Out}_G(L) \to \mathrm{Out}(L)$. Therefore by Theorem 5 there exists $A^D \in \mathscr{D}_{D_\Gamma}(N_G(D_\Gamma))$ for which the map in 7.7 is

$$N_G(D_\Gamma) \to N_G(A)/N_D(A) \to \mathrm{Out}_G(A) \to \mathrm{Out}(L).$$

Hence by 7.4.1 the kernel of this map is $DC_G(A)$, while it is evident from 7.7 that K is contained in the kernel. Thus $K \leqslant DC_G(A)$, so as $D \leqslant K$ we have

(7.8) $K = DC_K(A).$

$D_\Gamma = AC_{D_\Gamma}(L)$, so $A^D = A^{D_\Gamma} = A^{C_{D_\Gamma}(L)}$. Then as $N_G(D_\Gamma)$ acts on A^D, by a Frattini argument $N_M(L)\, C_{D_\Gamma}(L) = UC_{D_\Gamma}(L)$ for some subgroup U of $N_G(A)$. As $C_{D_\Gamma}(L) \cap N(A) = 1$, U is a complement to $C_D(L)$ in $N_M(L)\, C_D(L)$.

Let $V = UC_K(A)$. Then $V \leqslant N(A)$. $C_M(L)\, C_D(L) = C(L) \cap N_M(L)\, C_D(L) = C(L) \cap UC_D(L) = C_U(L)\, C_D(L)$, so $DC_U(L) = DC_M(L) = E$. As K is a complement to $N_G(L)/E$, $N_K(L) = E$. Therefore:

(7.9) $N_K(L) = DC_U(L)$.

$N_G(L) = DN_M(L)$ and $N_M(L)\, C_D(L) = UC_D(L)$, so:

(7.10) $N_G(L) = UD$.

Therefore $N_K(L) = K \cap UD = C_K(A)D \cap UD$ by 7.8. Then by 7.9:

(7.11) $DC_U(L) = UD \cap C_K(A)D$.

U normalizes A and L and hence commutes with the projection $A \to L$, so $C_U(L) = C_U(A)$. By 7.9, $C_U(L) \leqslant K$, so:

(7.12) $C_U(L) = U \cap C_K(A)$.

From 7.11 and 7.12 we obtain:

(7.13) $(U \cap C_K(A))D = UD \cap C_K(A)D$.

Let $\alpha: V \to VD/D$ be the natural map. If $u \in U$ and $k \in C_K(A)$ with $uk \in \ker(\alpha)$ then $u\alpha = (k\alpha)^{-1} \in U\alpha \cap C_K(A)\alpha = (U \cap C_K(A))\alpha$ by 7.13. So as $U \cap D = 1$, $u \in C_K(A)$. Thus $uk \in C_K(A)$ so we have shown:

(7.14) $\ker(\alpha) \leqslant C_K(A)$.

Therefore $V \cap D = C_K(A) \cap D = C_D(A) = C_D(D_\Gamma)$. Also by hypothesis $N_G(D_\Gamma) = KN_G(L)$, while $KN_G(L) = KUD$ by 7.10, so by 7.8, $N_G(D_\Gamma) = DUC_K(A) = DV$. Hence:

(7.15) *$V/C_D(D_\Gamma)$ is a complement to $D/C_D(D_\Gamma)$ in $N_G(D_\Gamma)/C_D(D_\Gamma)$.*

As a consequence of 7.10 and 7.15 we have:

(7.16) $UC(D_\Gamma) = N_V(L)$.

From 7.16, $N_V(L)\, C_D(L) = UC_D(L)$, while $UC_D(L) = N_M(L)\, C_D(L)$ by construction. Thus $N_V(L)\, C_D(L) = N_M(L)\, C_D(L)$, so by 7.15 and Theorem 2, $N_M(D_\Gamma)\, C_D(D_\Gamma)$ is conjugate to V in $N_G(D_\Gamma)$. Therefore $N_M(D_\Gamma)$ acts on some member of A^D, and hence we may assume without loss of generality that $N_M(D_\Gamma) \leqslant N_G(A)$. Hence M acts on $B = \langle A^M \rangle$ and $B \in \mathscr{D}_D$. Thus $B \in \mathscr{D}(G)$ and $M \leqslant N_G(B) \in \varkappa^*_{2,\Gamma}$.

This completes the analysis of case (b), so during the rest of the proof of Theorem 7.5 we may assume (a) holds. Let $K = C_G(L)$, $X = N_M(L)\,C_D(L)$, and $Y = K \cap X$. As $M \in \mathscr{N}_1'$, X is a complement to L in $N_G(L)$ with $N_G(L) = XK$. By hypothesis $E \trianglelefteq N_G(D_\Gamma) \geqslant N_G(L)$, so $Y = E \cap K \trianglelefteq N_G(L)$. We now apply 7.3 with $N_G(L)$ in the role of H, observing Y is in the set Γ of 7.3 because $\mathrm{Aut}_G(L) \cong X/Y$ (as $M \in \mathscr{N}_1'$). It follows that:

(7.17) $Y \trianglelefteq N_G(L)$, $LK/Y = (LY/Y) \times (K/Y)$ *with the factors* $N_G(L)$-*isomorphic, and* $(X \cap LK)/Y$ *is a* $N_M(L)$-*invariant full diagonal subgroup.*

Notice $Y = C_D(L)\,C_M(L)$, $LY = E$, and $LK = DC_G(L)$. By hypothesis LK and E are normal in $N_G(D_\Gamma)$. Let $Z = N_G(D_\Gamma) \cap C(LK/E)$. Then $N_G(L) \leqslant N_G(Z)$ and as $N_G(D_\Gamma)$ is primitive on Γ, either $Z \leqslant N_G(L)$ or $N_G(D_\Gamma) = ZN_G(L)$. $L \cong LK/E = F^*(N_G(L)/E)$, so $Z \cap N_G(L) = E$, and hence in the latter case Z is a normal complement to $N_G(L)/E$ in $N_G(D_\Gamma)$. But then we are back in case (a), so we may assume $Z \leqslant N_G(L)$. Hence:

(7.18) $E = N_G(D_\Gamma) \cap C(LK/E)$.

By 7.17, L is $N_G(L)$-isomorphic to LK/E, so the map

(7.19) $N_G(D_\Gamma) \to \mathrm{Out}_{N(D_\Gamma)}(LK/E) \to \mathrm{Out}(L)$

when restricted to $N_G(L)$ is equal to the conjugation map $N_G(L) \to \mathrm{Out}(L)$. Therefore by Theorem 5 there is a class $A^D \in \mathscr{D}_{D_\Gamma}(N_G(D_\Gamma))$ such that the kernel of the map in 7.19 is the same as the kernel of the map $N_G(D_\Gamma) \to \mathrm{Out}_{N(A)}(A) \to \mathrm{Out}(L)$. By 7.18 the kernel of the map in 7.19 is E, while by 7.4.1 the kernel of the second map is $DC_G(A)$.

Thus we have shown:

(7.20) $E = DC_G(A)$.

As in case (a), there is a complement U to $C_D(L)$ in X which acts on A. Let $U_0 = U \cap LK$. Then:

(7.21) $X \cap LK = U_0 C_D(L)$.

We conclude from 7.17 and 7.21 that:

(7.22) $U_0 Y/Y$ *is a* $N_M(L)$-*invariant full diagonal subgroup of* $LK/Y = LY/Y \times K/Y$.

$LK = A(X \cap LK)$ by 7.22, since $LY = AY$. Thus

$$\begin{aligned} N_{LK}(A) &= (A(X \cap LK)) \cap N(A) \\ &= A(N(A) \cap X \cap LK) \\ &= A(N(A) \cap U_0 C_D(L)) \qquad \text{by 7.21} \\ &= A U_0 C_D(D_\Gamma). \end{aligned}$$

Let $V = U_0 C_D(D_\Gamma)$. We have shown:

(7.23) $N_{LK}(A) = AV$.

$C_G(A) \leqslant E \leqslant N_G(L)$ by 7.20, so the projection $A \to L$ commutes with $C_G(A)$ and thus:

(7.24) $C_G(A) \leqslant C_G(L) = K$.

Next $N_{LK}(A)$ induces inner automorphisms on A, so $N_{LK}(A) \leqslant AC_G(A)$, and the opposite inclusion follows from 7.24, so:

(7.25) $N_{LK}(A) = A \times C_G(A)$.

Let $a: AC_G(A) \to A$ be the projection map. For $v \in V$, $v = a(v)k$ for some $k \in K$ by 7.24, so as $LY = AY$ we conclude:

(7.26) *$Ya(v)$ is the projection of Yv on LY/Y for each $v \in V$.*

Let us examine the map in 7.19 more closely. Given an isomorphism $\alpha: T \to S$ let $\alpha^*: \mathrm{Out}(T) \to \mathrm{Out}(S)$ be the induced isomorphism defined by $\alpha^*(\mu_T \circ \varphi) = \mu_S \circ \alpha \circ \varphi \circ \alpha^{-1}$, for $\varphi \in \mathrm{Aut}(T)$ and $\mu_T: \mathrm{Aut}(T) \to \mathrm{Out}(T)$ and $\mu_S: \mathrm{Aut}(S) \to \mathrm{Out}(S)$ the natural maps. We chose A so that the diagram

$$\begin{tikzcd}
& \mathrm{Out}_{N(A)}(A) \arrow[dr, "\pi^*"] & \\
N_G(A) \arrow[ur] \arrow[dr] & & \mathrm{Out}(L) \\
& \mathrm{Out}_{N(A)}(LK/E) \arrow[uu, dashed, "\theta^*"] \arrow[ur, "\sigma^*"'] &
\end{tikzcd}$$

commutes, where $\pi: A \to L$ is the projection and (using 7.21) $\sigma: LK/E \to L$ is the composition of the inverse of the projection $V/Y \to K/Y \cong LK/E$ with the projection $V/Y \to LY/Y \cong L$. Let $\theta = \sigma \circ \pi^{-1}$; then we can fill in θ^* in the diagram and still keep it commutative. Observe that by 7.26 and the definition of θ:

(7.27) *$\theta(Ev) = a(v)$ for each $v \in V$.*

Let $y \in N_G(A)$. Then y acts on $AC_G(A) = AV$ by 7.23 and 7.25. Thus for $v \in V$, $v^y = \gamma(v)\,\beta(v)$ for some $\gamma(v) \in A$ and $\beta(v) \in V$. Notice $(Ev)^y = E\beta(v)$. Moreover from the commutativity of the diagram, there exists $x \in A$ such that $\theta((Ev)^{yx}) = \theta(Ev)^{yx}$ for each $v \in V$. As $x \in A \leqslant E$, $(Ev)^{yx} = (E\beta(v))^x = E\beta(v)$, so by 7.27, $a(\beta(v)) = a(v)^{yx} = a(v^{yx})$ for all $v \in V$.

Next $v^{yx} = (\gamma(v)\,\beta(v))^x = \gamma(v)^x\,[x, \beta(v)^{-1}]\,\beta(v)$ with $\delta(v) = \gamma(v)^x\,[x, \beta(v)^{-1}] \in A$. So $a(\beta(v)) = a(v^{yx}) = a(\delta(v)\,\beta(v)) = \delta(v)\;a(\beta(v))$, and hence $\delta(v) = 1$. So $v^{yx} = \beta(v) \in V$, and therefore:

(7.28) $N_G(A) \leqslant N_G(V)A$.

Let $W = N_G(A) \cap N(V)$. By 7.28, $N_G(A) = WA$. As $LY = AY$, 7.22 implies $N_A(V) = 1$, so W is a complement to A in $N_G(A)$. Thus as $N_G(D_\Gamma) = N_G(A)\,D_\Gamma$, we conclude:

(7.29 *$W/C_D(D_\Gamma)$ is a complement to $D/C_D(D_\Gamma)$ in $N_G(D_\Gamma)/C_D(D_\Gamma)$.*

Recall that M acts on A and observe that U acts on U_0, and hence also on V. So $U \leqslant W$. Thus $C_D(L)\,N_W(L) = C_D(L)U = X = C_D(L)\,N_M(L)$, so applying Theorem 2 to $N_G(D_\Gamma)/C_D(D_\Gamma)$ and using 7.29, we conclude that $N_M(D_\Gamma)$ is conjugate in $N_G(D_\Gamma)$ to a subgroup of W. Hence without loss $N_M(D_\Gamma) \leqslant W$. Then M normalizes the direct product T of the M-conjugates of A, so $M \leqslant N_G(T) \in \mathscr{M}^*_{2,\Gamma}$, completing the proof of Theorem 7.5.

We are now in a position to prove Theorem 4, and hence complete the proof of Theorem 1. Let $M \in \mathscr{M}'_1$. As we observed during the proof of 7.2, $M \in \mathscr{M}^*_1$ if and only if M is contained in no member of $\mathscr{M}^*_2$. Hence as $\mathscr{M}^*_2$ is partitioned by $(\mathscr{M}^*_{2,\Gamma} : \Gamma \in \mathscr{P}^*(G, L))$, $M \in \mathscr{M}^*_1$ if and only if for each $\Gamma \in \mathscr{P}^*(G, L)$, M is contained in no member of $\mathscr{M}^*_{2,\Gamma}$. Therefore Theorem 7.5 implies part (1) of Theorem 4.

Let $H = N_G(L)$ and $\pi : H \to H/C_D(L)$ the natural map. By Theorem 2, $\mu : M \to N_M(L)\pi$ is a bijection of $\mathscr{M}'_1$ with the set $\mathscr{S}$ of complements $Y\pi$ to $L\pi$ in $H\pi$ with $\mathrm{Inn}(L) \leqslant \mathrm{Aut}_Y(L)$, and μ induces a bijection $\mu^* : \mathscr{C}'_1 \to \mathscr{C}$ of the corresponding orbits of G and H. By part (1) of Theorem 4, $\mu(\mathscr{M}^*_1)$ consists of those $Y\pi \in \mathscr{S}$ with $LC_Y(L) \in \mathscr{E}$. Adopt the notation of 7.3 with $(H\pi, L\pi)$ in the role of (H, L) in that lemma. Further choose notation so that $\mathscr{A}_i$ is nonempty if and only if $1 \leqslant i \leqslant m$. Observe that $\{LH_i : 1 \leqslant i \leqslant m\}$ is the set of normal subgroups of H satisfying conditions (i) and (ii) of the definition of $\mathscr{E}$ and $LC_Y(L) = LH_i$ for each $(Y\pi)^H \in \mathscr{A}_i$. ($H_i$ is defined in 7.3.) Therefore 7.3.3 implies part (2) of Theorem 4.

This completes the proof of Theorem 4.

We close this section with another parameterization of $\mathscr{C}^*_1$ and $\mathscr{C}'_1$. Namely, in the notation of Section 2, we prove:

(7.30) *Let $\mathscr{A}$ be the set of classes N^L of complements N to L in $N_G(L)/C_D(L)$ with $\mathrm{Inn}(L) \leqslant \mathrm{Aut}_N(L)$, let $\eta : \mathscr{C}' \to \mathscr{A}$ be the map supplied by Theorem 2, let $\eta : \mathscr{A} \to \mathscr{H}\mathrm{om}^*_{N_G(L)}(DC_G(L)/D, L)$ be the map supplied by 2.2, and let $\alpha = \eta \circ \mu$ be the composition of η and μ. Then*

(1) *α defines a bijection $\mathscr{C}'_1 \cong \mathscr{H}\mathrm{om}_{N_G(L)}(DC_G(L)/D, L)$.*

(2) *$\alpha(\mathscr{C}_1^*)$ consists of those orbits φ^L such that the inverse image in $DC_G(L)$ of* $\ker(\varphi)$ *is in $\mathscr{E}$.*

Proof. By Theorem 2 and 2.3, μ and η are bijections, so α is a bijection. By Theorem 4, $M^G \in \mathscr{C}'$ is in $\mathscr{C}^*$ if and only if $E = DC_M(L) \in \mathscr{E}$. But $E/D = \ker(\eta_N)$ where $N = N_M(L)C_D(L)/C_D(L)$, and η_N is defined in 2.2 and $N = \mu(M)$, so (2) holds.

APPENDIX: CORRECTIONS TO REF. [4]

In an appendix to [4] two theorems on maximal subgroups were stated and proved in outline. The following corrections should be made in the statements of these results: In the first theorem "Let H" at the beginning should read "Let $H \neq 1$," and the expression "prime cardinality" in part (a) should be "cardinality $\neq 1$." In part (e) of the second theorem "p" should be "k"; also "subgroup" in the first line should be "proper subgroup." Corrected statements of these theorems are given below.

THEOREM. *Let $H \neq 1$ be a subgroup of a finite direct product $G = \prod_{i \in I} G_i$ of isomorphic nonabelian simple groups. Then the transitive permutation representation of G on G/H extends to a primitive permutation representation of some group in which G is the socle if and only if either*

(a) *there is a partition $\mathscr{P}$ of I into subsets of equal cardinality $\neq 1$ with H the direct product $\prod_{S \in \mathscr{P}} \Delta_S$ of full diagonal subgroups Δ_S of the subproducts $\prod_{i < S} G_i$, or*

(b) *the subgroup H is a direct product $\prod_{i \leqslant I} H_i$ where H_i is a subgroup of G_i which is an intersection $G_i \cap \tilde{H}_i$ for some maximal subgroup $\tilde{H}_i$ of a group $\tilde{G}_i$ with $G_i \subseteq \tilde{G}_i \subseteq \operatorname{Aut} G_i$. Also, for each pair i, j of indices there must be an isomorphism of G_i with G_j carrying H_i to H_j.*

THEOREM. *Let $M \neq \mathscr{A}_n$ be a proper subgroup of the symmetric group S_n. Then some conjugates of M is contained in one of the subgroups listed below. Here $1 < m < n$ and p is prime.*

(a) $S_m \times S_k$, *$m + k = n$ (intransitive),*

(b) $S_m \operatorname{wr} S_k$, *$mk = n$ (imprimitive),*

(c) $S_m \operatorname{wr} S_k$, *$m^k = n$, $m \geqslant 5$ (product action),*

(d) $V.GL(V)$, *$p^k = n = |V|$; V a vector space over $GF(p)$,*

(e) $(G \operatorname{wr} S_k)\operatorname{Out} G$, *$|G|^{k-1} = n$, G a nonabelian simple group,*

(f) *an automorphism group of a nonabelian simple group $G < \mathscr{A}_n$, containing G and acting primitively (the full normalizer of G in S_n).*

The group in (e) is the extension of $G \operatorname{wr} S_k$ by the outer automorphism group $\mathrm{Out}(G)$ obtained from the natural extension $\mathrm{Aut}(G)$ of a diagonal copy of G; this isotropy group is $\mathrm{Aut}(G) \times S_k$. As mentioned in [4] this theorem was obtained independently by Mike O'Nan.

The assumption $H \neq 1$ in the first theorem is necessary, as the examples at the end of the section Main Results show. A similar correction should be made in Cameron's statement of this result. Theorem 4.1 in [1]. (The possibility $G_\alpha \cap N = 1$ should be allowed.) However, this probably does not affect the applications he discusses. It has no effect on the list above of possible maximal subgroups of the symmetric and alternating groups, since the groups arising this way are contained in subgroups of type (c).

References

1. P. Cameron, Finite simple groups and finite permutation groups, *J. London Math. Soc.* **13** (1981), 1–22.
2. D. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associative Algebras," Wiley, New York, 1962.
3. S. MacLane, "Homology," 3rd corrected ed., Springer-Verlag, New York, 1975.
4. L. Scott, Representations in characteristic p, *Proc. Sympos. Pure Math.* **37** (1980), 319–332.
5. J. Serre, "Corps locaux," Hermann, Paris, 1962.
6. A. Borel and J. Serre, Theoremes des definitude en cohomologie galoisienne, *Comm. Math. Helv.* **39** (1964), 111–163.
7. F. Gross and L. Kovacs, On normal subgroups which are direct products, preprint.
8. F. Gross and L. Kovacs, Maximal subgroups in composite finite groups, preprint.